1. Continuous Functions

The Intermediate Value Theorem

Let $f$ be a continuous function on an interval $[a, b]$. Then for every number $c$ between $f(a)$ and $f(b)$, there is some $x \in [a, b]$ where $f(x) = c$.

Let’s say that a function $f$ defined on an interval $[a, b]$ has the Intermediate Value Property on $[a, b]$ if for every subinterval $[\alpha, \beta] \subseteq [a, b]$ and every number $c$ between $f(\alpha)$ and $f(\beta)$, there is some $x \in [\alpha, \beta]$ where $f(x) = c$.

The Intermediate Value Theorem says that all continuous functions on $[a, b]$ have the intermediate value property on $[a, b]$. 
Are there *discontinuous* functions on \([a, b]\) that have the intermediate value property on \([a, b]\)?

Functions with “jump discontinuities” don’t. But some functions have another, *nastier* kind of discontinuity.

Consider the function

\[
f(x) = \begin{cases} 
\sin(1/x) & \text{if } x \neq 0 \\
0 & \text{if } x = 0
\end{cases}
\quad \text{on } [-1, 1].
\]
2. Differentiable Functions

All differentiable functions are continuous, but not all continuous functions are differentiable.

The Mean Value Theorem

Corollaries

If \( f'(x) = 0 \) for all \( x \in (a, b) \), then \( f(x) \) is constant on \( (a, b) \).

If \( f'(x) = g'(x) \) for all \( x \in (a, b) \), then \( f(x) - g(x) \) is constant on \( (a, b) \).
The Fundamental Theorem of Calculus

If $f$ is continuous on an interval $[a, b]$, then for all $x \in [a, b]$,

$$\frac{d}{dx} \int_{a}^{x} f(t) \, dt = f(x).$$

In other words...

Every continuous function $f$ on an interval $[a, b]$ has an antiderivative—namely,

$$\int_{a}^{x} f(t) \, dt.$$

In still other words...

Every continuous function $f$ on an interval $[a, b]$ is the derivative of some function on $[a, b]$. 


**But what about all that \( F(b) - F(a) \) business?**

Combine the theorem above with a corollary to the MVT:

Any two antiderivatives of a function \( f \) on an interval \( I \) differ by a constant on \( I \). That is, if \( f'(x) = g'(x) \) for all \( x \in I \), then \( f(x) - g(x) \) is constant on \( I \).

Here’s how it goes:

Suppose that \( f \) is continuous on \([a, b]\), and that \( F(x) \) is any antiderivative of \( f \) on \([a, b]\) that you happen to know. Then

\[
\int_a^x f(t) \, dt - F(x) = C
\]

for all \( x \in [a, b] \). The constant \( C \) has to equal \(-F(a)\). So,

\[
\int_a^x f(t) \, dt = F(x) - F(a)
\]

for all \( x \in [a, b] \), and in particular,

\[
\int_a^b f(t) \, dt = F(b) - F(a).
\]

**Inquiring minds want to know...**

The FTC tells us that every continuous function \( f \) on an interval \([a, b]\) is the derivative of some function on \([a, b]\). But...

**Are there derivatives that aren’t continuous?**

That is, are there functions \( f \) on \([a, b]\) such that \( f'(x) \) exists on \([a, b]\) but \( f' \) is not continuous on \([a, b]\)?
A derivative can’t have a jump discontinuity unless it’s undefined at the jump. Why?

But can a derivative have a “nasty” discontinuity?

Consider this:

\[ f(x) = \begin{cases} 
  x^2 \sin(1/x) & \text{if } x \neq 0 \\
  0 & \text{if } x = 0 
\end{cases} \quad \text{on } [-1, 1]. \]
Here’s a better view of what’s happening near $x = 0$.

And this is the derivative:

$$f'(x) = \begin{cases} 
2x \sin(1/x) - \cos(1/x) & \text{if } x \neq 0 \\
0 & \text{if } x = 0 
\end{cases}$$

So $f'(0)$ exists, while $f'$ is not continuous at $x = 0$. 
A Theorem.

A derivative can be discontinuous at a point where it exists, but it will have the intermediate value property on any interval on which it exists.

So, while a derivative needn’t be a continuous function, it does share one important property with every continuous function.
What statement does the following example prove false?

\[
f(x) = \begin{cases} 
  x^4 (2 + \sin(1/x)) & \text{if } x \neq 0 \\
  0 & \text{if } x = 0 
\end{cases}
\]

\[
f'(x) = \begin{cases} 
  4x^3 (2 + \sin(1/x)) - x^2 \cos(1/x) & \text{if } x \neq 0 \\
  0 & \text{if } x = 0 
\end{cases}
\]
What statement does this example prove false?

\[ f(x) = \begin{cases} 
  x + 2x^2 \sin(1/x) & \text{if } x \neq 0 \\
  0 & \text{if } x = 0 
\end{cases} \quad \text{on } [-1, 1]. \]

\[ f'(x) = \begin{cases} 
  1 + 4x \sin(1/x) - 2 \cos(1/x) & \text{if } x \neq 0 \\
  0 & \text{if } x = 0 
\end{cases} \]
The FTC revisited

Every continuous function $f$ on an interval $[a, b]$ has an antiderivative—namely,

$$\int_a^x f(t) \, dt.$$  

Often, such a function has no simpler form. Common examples are known as “special functions.”

Example 1

$$\int_1^x \frac{e^t}{t} \, dt$$

$$f(x) = \int_1^x \frac{e^t}{t} \, dt$$

$$= -\text{ExpIntegralEi}[1] + \text{ExpIntegralEi}[x]$$

![Graph of the function](image)
Example 2

\[\int_0^x \sin t^2 \, dt\]

\[\sqrt{\frac{\pi}{2}} \text{FresnelS} \left[ \sqrt{\frac{2}{\pi}} x \right]\]

Example 3

\[\int_0^x e^{-t^2} \, dt\]

\[\frac{1}{2} \sqrt{\pi} \text{ Erf} [x]\]