1999 Math Tournament Solutions

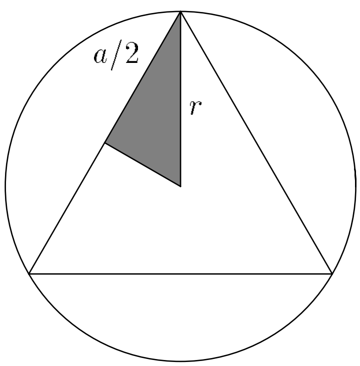
1. (A) The square root symbol, by definition, represents a number greater than or equal to zero.

2. (B) By the multiplication principle, the first three places can be assigned in $10 \cdot 9 \cdot 8 = 720$ ways.

3. (D) The mean is $\frac{3+5+6+12+14}{5} = 8$, the median is 6, and the variance is $\frac{5^2+3^2+2^2+4^2+6^2}{5} = 18$. Hence, the standard deviation is $3\sqrt{2}$. The product is $144\sqrt{2}$.

4. (E) Adding 2100 and 750 counts the students who took a math and a psych twice. Therefore, we must subtract 300 from this total to get the true number, 2550. Note that the logical meaning of the word *or*, in this case, includes those that took both subjects.

5. (C) Consider the right triangle formed as below. Since one of the angles bisects the interior angle of the equilateral triangle, its measure is 30°. Therefore, $\cos 30° = \frac{a/2}{r} = \frac{a}{2r}$. Since, $\cos 30° = \frac{\sqrt{3}}{2}$, solving for r, we get that $r = \frac{a}{\sqrt{3}}$. Hence, the area of the circle is $\frac{\pi a^2}{3}$.



6. (B) There are 50 integers divisible by 2, 33 integers divisible by 3, and 16 integers divisible by 6 (divisible by both 2 and 3). Therefore, there are $50 + 33 - 16 = 67$ integers divisible by 2 or 3.

7. (E) The locus of points is the line which is perpendicular to and passes through the midpoint of the segment joining the two points.

8. (C) The two triangles are similar (use the fact that vertical angles are equal and that opposite interior angles are equal). Lengths of similar figures are proportional, but the areas of two similar figures are proportional to the lengths squared; that is, if we let $A =$ area of $\triangle APB$, then $A/32 = 36/64$. Therefore, $A = 18$.

9. (C) The perimeter is obtained by adding the lengths of the three sides. The area of a right triangle can be obtained by taking one-half of the product of the two shorter sides. Using these criteria we can rule out all but the answer $24, 25, 7$. As a final check, we see that these numbers are a Pythagorean Triple.

10. (D) There are 14 perfect squares between 1 and 200 inclusive (note that $14^2 = 196$ is the largest perfect square less than or equal to 200). Therefore, the probability is $14/200 = .07$.

11. (B) Since no three points are colinear, any three points of the six may be used as the vertices of a triangle. Therefore, simply find the number of three-element subsets of a six-element set; that is, $\binom{6}{3} = 20$.

12. (C) From the equation of the given line, we have that $y = 2 - x$. Substituting this expression for y in the equation of the circle yields

$$\begin{aligned} x^2 + (2-x)^2 &= 10 \\ 2x^2 - 4x - 6 &= 0 \\ 2(x^2 - 2x - 3) &= 0 \\ 2(x-3)(x+1) &= 0. \end{aligned}$$

Therefore, $x = 3$ or -1.

13. (C) In traveling from the lower-left corner and moving clockwise around the figure to the upper-right corner, we travel a total of 4 units to the right, backtracking one unit, for a total of 6 units traveled horizontally. We travel 6 units vertically to get to the upper-right corner, for a total of 12 units from the lower-left corner clockwise to the upper-right corner. It is easy to see that we travel 10 additional units to complete our journey. Therefore, the perimeter is 22 units.

14. (D) For each set of five cards chosen, that set could have been drawn in 5! different orders. Only one of those orderings (for each set of 5 cards) will be in increasing order. Therefore, the probability is 1/5!.

15. (C) Note that all of the choices are rational numbers. Any rational root of a polynomial with integer coefficients must pass the rational zero test; that is, it must be of the form $\dfrac{\text{factor of constant term}}{\text{factor of leading coefficient}}$. Among the possible choices, only $\frac{7}{3}$ satisfies this condition.

16. (B) Putting the unshaded parts together yields a square of area 9. Therefore, the shaded area covers 16 square units. Hence, the proportion that is shaded is $16/25 = 64\%$.

17. (B) This expression contains each reciprocal pair of trig functions. Therefore, the product is 1. The identity $\tan^2 x + 1 = \sec^2 x$ implies that $\sec^2 x - \tan^2 x = 1$. None of the other expressions is necessarily equal to one.

18. (A) The graph of a fourth degree polynomial has at most three turns. Graph "E" is a polynomial of odd degree, therefore, graph "A" is the only possible choice.

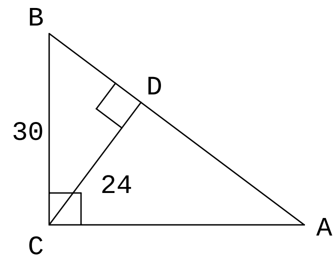
19. (C) The ball first travels 10 ft down; thereafter it travels both up and down a distance nine-tenths of the height of the previous bounce. The total distance is thus $10 + 2\left[9 + 9(0.9) + 9(0.9)^2 + \cdots\right]$, where the sum in brackets is a geometric series with common ratio $\frac{9}{10}$. The total distance is thus $10 + 2\frac{9}{1-0.9} = 10 + 18(10) = 190$ feet.

20. (C) In base eight, $777 = 1000 - 1$ and $7777 = 10,000 - 1$, so the product is $(1000-1)(10,000-1) = 10,000,000 - 10,000 - 1000 + 1 = 7,767,001$.

21. (E) There are 90 two-digit numbers (from 10 to 99). If ab is the decimal representation for a two-digit number n, then $n = 10a + b$, where a and b are digits and $a \geq 1$. If n is 4 times the sum of its digits, then $n = 10a + b = 4(a+b)$ and $b = 2a$. That means n must be 12, 24, 36, or 48. The probability is thus $\frac{4}{90} = \frac{2}{45}$.

22. (C) The two *longest* altitudes of a right triangle are simply the two legs. Thus, the shorter leg has length 30 cm, and the altitude to the hypotenuse is the shortest altitude, which has

length 24 cm. In the figure below, $BD = \sqrt{30^2 - 24^2} = 18$. Triangles ABC and CBD are



similar, so $\frac{AC}{30} = \frac{24}{18}$ and $AC = 40$ cm.

23. (A) $\cos 2\theta = 1 - 2\sin^2\theta = 1 - 2\left(\frac{2}{3}\right)^2 = \frac{1}{9}$.

24. (D) Draw a line from the center of the square down to the midpoint of the bottom edge. The shaded area is one-fourth the area of a circle with radius $\frac{1}{2}$ minus the area of an isosceles right triangle with legs of length $\frac{1}{2}$. That makes $\frac{1}{4}\pi(\frac{1}{2})^2 - (\frac{1}{2})(\frac{1}{2})(\frac{1}{2}) = \frac{\pi}{16} - \frac{1}{8} = \frac{\pi-2}{16}$.

25. (C) Subtracting $\sqrt{x}$ from both sides gives $0 = (1-x)\sqrt{x}$, which has two real solutions: $x = 0$ and $x = 1$.

26. (D) The parabola will be tangent to the x-axis if the vertex has a y-coordinate of 0. Putting the equation in standard form, we get $y = 2(x + \frac{1}{2})^2 + (k - \frac{1}{2})$, so $k = \frac{1}{2}$.

27. (B) First dividing by C, $\frac{D}{C} = (AB)^{\frac{1}{n}}$. Taking the log of both sides gives $\log D - \log C = \dfrac{\log A + \log B}{n}$ and $n = \dfrac{\log A + \log B}{\log D - \log C}$.

28. (D) $g(x) = f(x-3) + 2 = 2(x-3)^2 - 3(x-3) + 1 + 2 = 2x^2 - 15x + 30$.

29. (B) First note that $\overline{DE}$ is parallel to $\overline{BC}$ and half as long, so that $\triangle ADE$ is equilateral and DEFG is a rectangle. Let H be the midpoint of $\overline{DE}$. Then triangles AHE, AHD, EFC, and DGB are all congruent. If $DE = x$ and $EF = y$, then the area of the rectangle DEFG is xy, while the area of $\triangle ABC$ is $\frac{1}{2}(2x)(2y) = 2xy$, so the ratio is $\frac{1}{2}$.

30. (C) Since $\triangle ADE$ is a 30-60-90 triangle, $AD = 2\sqrt{3}$. Thus the total area is $6 = 2\sqrt{3} + 2\sqrt{3}(AB) = 2\sqrt{3}(1 + AB)$ and $AB = \sqrt{3} - 1$.

31. (B) Let x be the number of cm by which the length and the width are reduced. Then, since the new volume is 72% of the old volume,

$$\begin{aligned} (10-x)(5-x)2 &= (0.72)(10)(5)(2) = 72 \\ x^2 - 15x + 50 &= 36 \\ (x+14)(x-1) &= 0 \end{aligned}$$

Therefore, x must be 1 cm, and the new length is 9 cm.

32. (B) If Andi is the perpetrator or if none of the four suspects is the perpetrator, then Bob and Dave are telling the truth. If Bob is the perpetrator, then only Dave is telling the truth. If Carla is the perpetrator, then Andi, Bob and Dave are telling the truth. If Dave is the perpetrator, then Bob and Carla are telling the truth. Since exactly one suspect is telling the truth, Bob committed the crime.

33. (B) The last 4 digits must be either all ones, all twos, or all fours. There are 10^4 possibilities, so the probability is $\frac{3}{10^4} = 0.0003$.

34. (D) Let x represent the original number of rows; each of these rows has $\frac{72}{x}$ seats. With three more seats per row and two fewer rows, there are still 72 seats, giving

$$\begin{aligned}(x-2)(\frac{72}{x}+3) &= 72\\ (x-2)(3x+72) &= 72x\\ (x-2)(x+24) &= 24x\\ x^2-2x-48 &= 0\\ (x+6)(x-8) &= 0\end{aligned}$$

That means the original number of rows must be 8.

35. (B) Consider the triangle formed by the landmark and the family's two positions, at noon and at 1:00 p.m. This is an isosceles triangle with a 60° angle, so it must be an equilateral triangle. Since the family traveled 60 mph for one hour, the northern edge of the triangle must be 60 miles long. The length of the altitude from the landmark to the northern edge of this equilateral triangle is $\left(\frac{\sqrt{3}}{2}\right)60 = 30\sqrt{3}$.

36. (E) There are $2^4 = 16$ possible subsets of forgotten items, of which 15 contain at least one item. There are $C_{4,2} = 6$ subsets with 2 elements, so the probability is $\frac{6}{15} = \frac{2}{5}$.

37. (D) Nine trips suffice. Assume they all begin on the left side of the lake. Label the men as A and B and the boys as a and b, a trip to the left as L and a trip to the right as R.
(1)a and b go R; (2)a goes L;
(3)A goes R; (4)b goes L;
(5)a and b go R; (6)a goes L;
(7)B goes R; (8)b goes L;
(9)a and b go R.

38. (D) The graph of f is reflected about the y-axis and compressed horizontally by a factor of 2.

39. (D) The total number of students is $16 + 32 + 35 + 25 + 12 = 120$. The proportion of A students is $\frac{12}{120} = \frac{1}{10}$. The central angle would then be $\frac{1}{10}360° = 36°$.

40. (A) From the quadratic formula the two roots are $\dfrac{-p \pm \sqrt{p^2-4q}}{2}$ and the sum of their reciprocals is $\dfrac{2}{-p+\sqrt{p^2-4q}} + \dfrac{2}{-p-\sqrt{p^2-4q}} = \dfrac{2(-p-\sqrt{p^2-4q})+2(-p+\sqrt{p^2-4q})}{(-p+\sqrt{p^2-4q})(-p-\sqrt{p^2-4q})} = \dfrac{-4p}{p^2-(p^2-4q)} = \dfrac{-4p}{4q} = -\dfrac{p}{q}$.
So, the reciprocals of the two
Roots of x^2+px+q
Set equal to zero
Add up to be, oh...
Negative p over q.