Solutions for Multiple Choice Test
High School Mathematics Tournament
Armstrong Atlantic State University, February 14, 1998

1. (B) $\dfrac{\frac{1}{2} - 3.75}{3\frac{1}{4} + \frac{13}{8}} = \dfrac{\frac{4}{8} - \frac{30}{8}}{\frac{26}{8} + \frac{13}{8}} = \dfrac{-\frac{26}{8}}{\frac{39}{8}} = -\frac{26}{8} \times \frac{8}{39} = -\frac{2}{3}$

2. (A) H = 3qN – 5. Substituting H = 10, q = 2 into the equation, we find $10 = 6N - 5 \Rightarrow 15 = 6N \Rightarrow N = \frac{5}{2}$. Now, replace N with $\frac{5}{2}$ to establish $H = \frac{15}{2}q - 5$. Let H = 25 and solve for q. $25 = \frac{15}{2}q - 5 \Rightarrow 30 = \frac{15}{2}q \Rightarrow q = 4$.

3. (B) $\sum_{i=1}^{5} (-1)^i (i + 2) = (-1)^1(1 + 2) + (-1)^2(2 + 2) + (-1)^3(3 + 2) + (-1)^4(4 + 2) + (-1)^5(5 + 2) = -3 + 4 - 5 + 6 - 7 = -5$

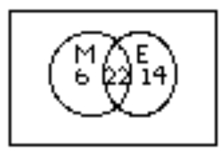
4. (C) Distance traveled = (3hr)(50mph) + (5hr)(62mph) = 150 miles + 310 miles = 460 miles. Time elapsed = 3 hr + 5 hr = 8 hr. Average speed = $\dfrac{\text{distance traveled}}{\text{time elapsed}} = \dfrac{460 \text{ miles}}{8 \text{ hr}} = 57\frac{1}{2}$ mph.

5. (B) Since (3,–5) lies on the graph of *l*, $a(3) + 7(-5) = 12 \Rightarrow a = \frac{47}{3}$. Thus the equation of *l* is $\frac{47}{3}x + 7y = 12$ or $y = -\frac{47}{21}x + \frac{12}{7}$. The slope of *l* is $-\frac{47}{21}$ so a line perpendicular to *l* has slope $\frac{21}{47}$.

6. (E) Let x = Brianna’s mother’s present age and x – 20 = Brianna’s present age. Then, $x - 10 = 2(x - 30) \Rightarrow x = 50$.

7. (D) Since plants are 18 inches apart in rows, 9 plants can be planted in each row. Rows must be 12 inches apart, so 13 rows can be planted. (9 plants/row)(13 rows) = 117 plants. (117 plants)(3 ears/plant) = 351 ears.

8. (A) Number in neither math nor English 50 – (6+22+14) = 8.



9. (D) Let x = sum of scores for class of 35 and y = sum of scores for class of 25. Then $\frac{x}{35} = 70$ and $\frac{y}{25} = 85 \Rightarrow x = 2450$, y = 2125. Mean grade for all students in both classes = $\frac{x + y}{35 + 25} = \frac{2450 + 2125}{60} = \frac{4575}{60} = 76.25$

10. (B) Functions generated are: {(1, 4), (2, 4), (3, 5)} / {(1, 4), (3, 4), (2, 5)} / {(2, 4), (3, 4), (1, 5)} / {(1, 5), (2, 5), (3, 4)} / {(1, 5), (3, 5), (2, 4)} / {(2, 5), (3, 5), (1, 4)}.

11. (E) Let x = liters of 25% solution used. Hence $.25x + .5(10 - x) = .3(10) \Rightarrow .25x + 5 - .5x = 3 \Rightarrow 2 = .25x \Rightarrow x = 8$.

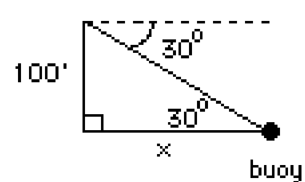
12. (B) $\frac{1}{x - \sqrt{y}} + \frac{1}{\sqrt{y} + x} = \frac{\sqrt{y} + x}{(x - \sqrt{y})(\sqrt{y} + x)} + \frac{x - \sqrt{y}}{(x - \sqrt{y})(\sqrt{y} + x)} = \frac{\sqrt{y} + x + x - \sqrt{y}}{(x - \sqrt{y})(\sqrt{y} + x)} = \frac{2x}{(x - \sqrt{y})(x + \sqrt{y})} = \frac{2x}{x^2 - y}$.

13. (E) If t = 0, then p = 0. For $0<t\leq 1$, p = \$.24. For $1< t \leq 2$, p = \$.48. For $2 < t \leq 3$, p = \$.72, etc.

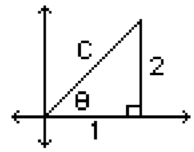
14. (A) Let b = base of original triangle and h = height of original triangle. Area of original triangle = $\frac{1}{2}bh$. New triangle has base = 1.1b and altitude = .9h. Area of new triangle = $\frac{1}{2}(1.1b)(.9h) = \frac{1}{2}(.99)bh$ which represents a 1% decrease in area.

15. (D) $\log_3 x = \log_y 5 = 2 \Rightarrow 3^2 = x$ and $y^2 = 5$. Hence, $(xy)^2 = x^2y^2 = (3^2)^2(5) = (81)(5) = 405$.

16. (A) Let x = distance in feet from buoy to base of lighthouse. Construct a right triangle to depict setting (see picture at right). Since $\tan 30^o = \frac{100}{x}$, $x = \frac{100}{\tan 30^o} = \frac{100}{\frac{1}{\sqrt{3}}} = 100\sqrt{3}$.



17. (A) Let $\theta = \arctan 2$, then $\tan\theta = 2$. Construct a right triangle with acute angle θ.
Since $\tan\theta = 2 = \dfrac{\text{length of edge opposite } \theta}{\text{length of edge adjacent to } \theta}$, label edges of triangle as pictured.
Find length of hypotenuse, c, using Pythagorean Theorem: $1^2 + 2^2 = c^2 \Rightarrow c = \sqrt{5}$.
Hence $\sec(\theta) = \dfrac{\text{length of hypotenuse}}{\text{length of edge adjacent to } \theta} = \dfrac{\sqrt{5}}{1} = \sqrt{5}$.



18. (B) To generate the graph of $y = g(x)$, reflect $y = f(x)$ about the y-axis then shift one unit upward. Hence, $g(x) = f(-x) + 1$.

19. (D) $125! = 125\cdot124\cdot123\cdots3\cdot2\cdot1$. Then there are $\frac{125}{5} = 25$ multiples of 5, $\frac{125}{25} = 5$ multiples of 25, and one (125) multiple of $125 = 5^3$. So 5^{31} divides 125! and 5^{32} does not divide 125!. 5^{31} is the maximum power of 5 that divides 125!.

20. (D) In decimal form $N = 125A + 2(5) + 3 = 125A + 13$. N is also $7^3 + 6(7) + A = 385 + A$. $\therefore$ $125A + 13 = 385 + A \Rightarrow 124A = 372 \Rightarrow A = 3$.

21. (C) Let x = the length of the diagonal of the 3″× 5″ rectangle and d = the length of the corresponding diagonal drawn in the enlargement (both in inches). By the Pythagorean Theorem, $3^2 + 5^2 = x^2 \Rightarrow 34 = x^2 \Rightarrow x = \sqrt{34}$. Since the triangle created by drawing the diagonal in the 3″× 5″ rectangle is similar to the triangle created in the enlargement, corresponding sides of the triangles are in proportion. Thus $\frac{3}{4} = \frac{\sqrt{34}}{d} \Rightarrow 3d = 4\sqrt{34} \Rightarrow d = \frac{4\sqrt{34}}{3}$.

22. (B) By the Pythagorean Theorem, $(DA)^2 + (AC)^2 = (DC)^2 \Rightarrow 5^2 + (AC)^2 = 13^2 \Rightarrow (AC)^2 = 144 \Rightarrow AC = 12$. Let x = BC so $12 - x = AB$ and $15 - x = DB$ (Note: $DB + BC + DC = 28$). By the Pythagorean Theorem, $(DB)^2 = (AD)^2 + (AB)^2 \Rightarrow (15 - x)^2 = 5^2 + (12 - x)^2 \Rightarrow x^2 - 30x + 225 = 25 + 144 - 24x + x^2 \Rightarrow 56 = 6x \Rightarrow x = \frac{28}{3}$.

23. (C) Area of Sue's slice $= \frac{1}{2}(5)^2\left(\frac{2}{15}\pi\right) = \frac{5}{3}\pi$. $\left(\text{Note: } 24^o = \frac{24}{180}\pi \text{ radians} = \frac{2}{15}\pi \text{ radians}\right)$ Area of Kathy's slice $= \frac{1}{2}(10)^2\left(\frac{\pi}{7}\right) = \frac{50\pi}{7}$. So, $\frac{50\pi}{7} = \mathbf{k}\left(\frac{5}{3}\pi\right) \Rightarrow \mathbf{k} = \frac{50\pi}{7}\cdot\frac{3}{5\pi} = \frac{30}{7}$.

24. (B) When $p(x) = 2x^3 - x^2 - 13x - 6$ is divided by $x + 2$, the quotient is $2x^2 - 5x - 3$ which in factored form is $(2x + 1)(x - 3)$. Hence $p(x) = (x + 2)(2x + 1)(x - 3)$ and zeros of p(x) are $-2, -\frac{1}{2}$ and 3. $\therefore$ $r_1 + r_2 = -\frac{1}{2} + 3 = \frac{5}{2}$.

25. (B) Ratio of new book budget to total library budget starting in 1987 : $\frac{50{,}000^*}{105{,}000}, \frac{55{,}000}{100{,}000}, \frac{50{,}000^*}{110{,}000}, \frac{55{,}000^*}{115{,}000}, \frac{60{,}000^*}{130{,}000}, \frac{70{,}000}{135{,}000}, \frac{80{,}000}{140{,}000}, \frac{85{,}000}{135{,}000}, \frac{80{,}000}{145{,}000}$. Four of these ratios (denoted with *) are less than 50%

26. (E) There are 6 × 6 or 36 equally likely possible outcomes. Fifteen of these outcomes will have a sum which is prime. Pairs that have of sum of 2: (1,1), a sum of 3: (1, 2), (2,1), a sum of 5: (1, 4), (4, 1), (2, 3), (3, 2), a sum of 7: (1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3), a sum of 11: (5, 6), (6, 5). The probability of a prime sum is $\frac{15}{36} = \frac{5}{12}$.

27. (D) f satisfies $f(x^2) = (x^2 + 1)\cdot f(x)$ and $f(2) = 3$. So, $f(256) = f(16^2) = (16^2 + 1)\cdot f(16) = 257\cdot f(16)$.
$f(16) = f(4^2) = (4^2 + 1)\cdot f(4) = 17\cdot f(4)$. $f(4) = f(2^2) = (2^2 + 1)\cdot f(2) = 5\cdot 3 = 15$. Hence, $f(16) = 17\cdot 15 = 255$ and $f(256) = 257\cdot 255 = 65{,}535$.

28. (B) Since trapezoid ABCD is isosceles, AD = BC and $m\angle DCB = m\angle CDA$. Hence, $\Delta ADC \cong \Delta BCD$ by SAS. Since corresponding parts are congruent, $\angle DCA \cong \angle CDB$ and DB = CA. Now, $\Delta DAB \cong \Delta CBA$ by SSS $\Rightarrow \angle CAB \cong \angle DBA$. Since GE is an altitude, $m\angle DGE = m\angle CGE = m\angle AEG = m\angle BEG = 90^o$. Hence, $\Delta CGF \cong \Delta DGF$ and $\Delta FEB \cong \Delta FEA$ by LA. By corresponding edges, DG = CG = 3 and AE = BE = 5. Since $DC \parallel AB$, $\angle DCA \cong \angle BAC$ by alternate interior angles. Hence, $\Delta CGF \sim \Delta AEF$. Corresponding parts of similar triangles are in proportion so $\dfrac{CG}{AE} = \dfrac{GF}{EF}$.
Let EF = x and GF = 8 − x. Thus $\dfrac{3}{5} = \dfrac{8 - x}{x} \Rightarrow 3x = 40 - 5x \Rightarrow x = 5$.

29. (C) The chart shows the possible combinations when choosing two teams from the three largest cities: New York (NY), Chicago (CH), or Los Angeles (LA) and the implications that follow.

NY	CH	$\Rightarrow$	SD	B	PH				
CH	LA	$\Rightarrow$	SD	B	PH	or	SD	B	PH
NY	LA	$\Rightarrow$	SD	B	PH				

NY	CH	LA	$\Rightarrow$	SD	B	PH

30. (A) $m(\overline{BX}) = m(\overline{BZ})$, and $m(\overline{CY}) = m(\overline{CZ})$ so perimeter $= m(\overline{AB}) + m(\overline{BC}) + m(\overline{AC}) = m(\overline{AB}) + m(\overline{BZ}) + m(\overline{CZ}) + m(\overline{AC}) = [m(\overline{AB}) + m(\overline{BX})] + [m(\overline{CY}) + m(\overline{AC})] = 12 + 12 = 24$.

31. (C) Consider the possible selections: White, Red, Blue, White, White, Red, Red, Blue. 8 selections needed for 3 pairs. If there are 8 socks, then there are either (a) at least two socks of each color or (b) at least four socks of at least one color. In the first case, there are 3 matching pairs (one of each color). In the second case, there are two matching pairs from the four of one color. There must be at least one additional matching pair from the remaining 4 socks.

32. (E) Coordinates for vertices of square are: upper right vertex $(x, 4)$, upper left vertex $(-x, 4)$, lower right vertex (x, x^2), lower left $(-x, (-x)^2)$. Horizontal edge of square has length $2x$, vertical edge of square has length $4 - x^2$. Since edge lengths must be the same, $2x = 4 - x^2 \Rightarrow x^2 + 2x - 4 = 0$. Using the quadratic formula we find $x = -1 + \sqrt{5}$. Using $2x$ as the length of an edge, the area of the square $= \left(2\left(-1 + \sqrt{5}\right)\right)^2 = 24 - 8\sqrt{5}$.

33. (D) Each triangle must contain 3 of the nine 9 points as vertices. The possible combinations of 9 points chosen 3 at the time is $\binom{9}{3} = \frac{9!}{3!\,6!} = \frac{9 \cdot 8 \cdot 7}{3 \cdot 2 \cdot 1} = 84$. But not all of these combinations will produce a triangle. The three points chosen must be **noncollinear**. Subtracting those collinear combinations, (three combinations along the vertical, three along the horizontal, and two along the diagonal = 8) we find the number of triangles formed is $84 - 8 = 76$.

34. (D) The number of integers between 1 and 1000 inclusive that are divisible by 6 is $\text{Int}(1000/6) = 166$. Of these 166 numbers, every third number is divisible by 9 and must be removed: $\text{Int}(166/3) = 55$. Every fifth number is divisible by 15 and must also be removed: $\text{Int}(166/5) = 33$. However, every 15th number is divisible by both 9 and 15 and was thus removed twice: $\text{Int}(166/15) = 11$. $\therefore$ Those integers that are divisible by 6 but neither 9 nor 15 is $166 - 55 - 33 + 11 = 89$.

35. (C) $9^x - 9^{x-1} = 24 \Rightarrow 9^{x-1}(9 - 1) = 24 \Rightarrow 9^{x-1}(8) = 24 \Rightarrow 9^{x-1} = 3 \Rightarrow (3^2)^{x-1} = 3^1 \Rightarrow 2x - 2 = 1 \Rightarrow x = \frac{3}{2}$.

36. (A) $f(x) = \frac{2x^2 + 4x + 11}{x^2 + 2x + 5} = 2 + \frac{1}{x^2 + 2x + 5}$. The minimum value of $x^2 + 2x + 5$ is at the vertex at $x = -\frac{b}{2a} = -\frac{2}{2} = -1$. $(-1)^2 + 2(-1) + 5 = 4$. Thus the greatest value of the function is $2 + \frac{1}{4} = \frac{9}{4}$.

37. (D) For the first digit in the two-digit expression, the digits from 1 to 9 appear 10 times each. For the second digit in the two-digit expression, the digits from 0 to 9 appear 9 times each. So,
sum $= 10(1 + 2 + 3 + \ldots + 9) + 9(0 + 1 + 2 + 3 + \ldots + 9) = 19(1 + 2 + \ldots + 9) = 19(45) = 855$.

38. (C) Since ΔABC is equilateral, $m\angle A = m\angle B = m\angle C = 60^o$. Since $l \parallel BC$, by corresponding angles, $m\angle ADE = m\angle ABC = 60^o$ and $m\angle AED = m\angle ACB = 60^o$. Thus, ΔADE is equilateral. Let x = AD=DE=AE. Note: altitude of $\Delta ADE = \frac{\sqrt{3}}{2}x$ and altitude of trapezoid DECB $= \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}x$. We are given that area of ΔADE (A_1) = area of trapezoid BDEC (A_2). $A_1 = \frac{1}{2} \cdot x \cdot \frac{\sqrt{3}}{2}x$ and $A_2 = \frac{1}{2}(1+x)\left(\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}x\right)$. Hence,

$\frac{1}{2} \cdot x \cdot \frac{\sqrt{3}}{2}x = \frac{1}{2}(1+x)\left(\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}x\right) \Rightarrow \frac{\sqrt{3}}{4}x^2 = \frac{\sqrt{3}}{4}(1+x)(1-x) \Rightarrow x^2 = 1 - x^2 \Rightarrow 2x^2 = 1 \Rightarrow x = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}$.

The altitude of $\Delta ADE = \frac{\sqrt{3}}{2}\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{6}}{4}$.

39. (B) The areas of the three pizzas in square inches are 36π, 64π, and 100π. If X,Y, and Z are the number of each size respectively, then 36X+64Y+100Z=236 where X,Y, and Z are integers. The list of possible solutions is:

(3,2,0) $\Rightarrow$ cost 3($8.00)+2($10.00)=$44.00
(2,1,1) $\Rightarrow$ cost 2($8.00)+1($10.00)+1($12.00)=$38.00
(1,0,2) $\Rightarrow$ cost 1($8.00)+2($12.00)=$32.00

difference between the maximum and minimum cost is $12.00.

40. (C). Let r = common ratio for the sequence. So, b = ar, c = ar^2, d = ar^3, (a, b) = (a, ar), and (c, d) = (ar^2, ar^3). The slope of the line passing through (a, b) and (c, d), $m_l = \frac{ar^3 - ar}{ar^2 - a} = \frac{ar(r^2-1)}{a(r^2-1)} = r$. Thus the slope of the line perpendicular to l is $m_k = -\frac{1}{r}$. The midpoint of the segment connecting (a, b) and (c, d) is $M = \left(\frac{a+ar^2}{2}, \frac{ar+ar^3}{2}\right)$. The equation for line k is $y - \frac{ar+ar^3}{2} = -\frac{1}{r}\left(x - \frac{a+ar^2}{2}\right)$. To find the y-intercept of this line, let x = 0. Thus, $y - \frac{ar+ar^3}{2} = -\frac{1}{r}\left(-\frac{a+ar^2}{2}\right)$

$\Rightarrow y - \frac{ar+ar^3}{2} = \frac{a+ar^2}{2r} \Rightarrow y = \frac{a+ar^2}{2r} + \frac{ar+ar^3}{2} \Rightarrow y = \frac{a+ar^2+ar^2+ar^4}{2r}$. Since $r = \frac{b}{a}$, $y = \frac{a+2ar^2+ar^4}{2r}$ can be expressed as $y = \frac{a+2a\left(\frac{b}{a}\right)^2 + a\left(\frac{b}{a}\right)^4}{2\left(\frac{b}{a}\right)} \Rightarrow y = \frac{a + \frac{2b^2}{a} + \frac{b^4}{a^3}}{2\left(\frac{b}{a}\right)}$. Multiplying numerator and denominator by a^3 we get

$y = \frac{a^4 + 2a^2b^2 + b^4}{2a^2b}$ or $y = \frac{(a^2+b^2)^2}{2a^2b}$.