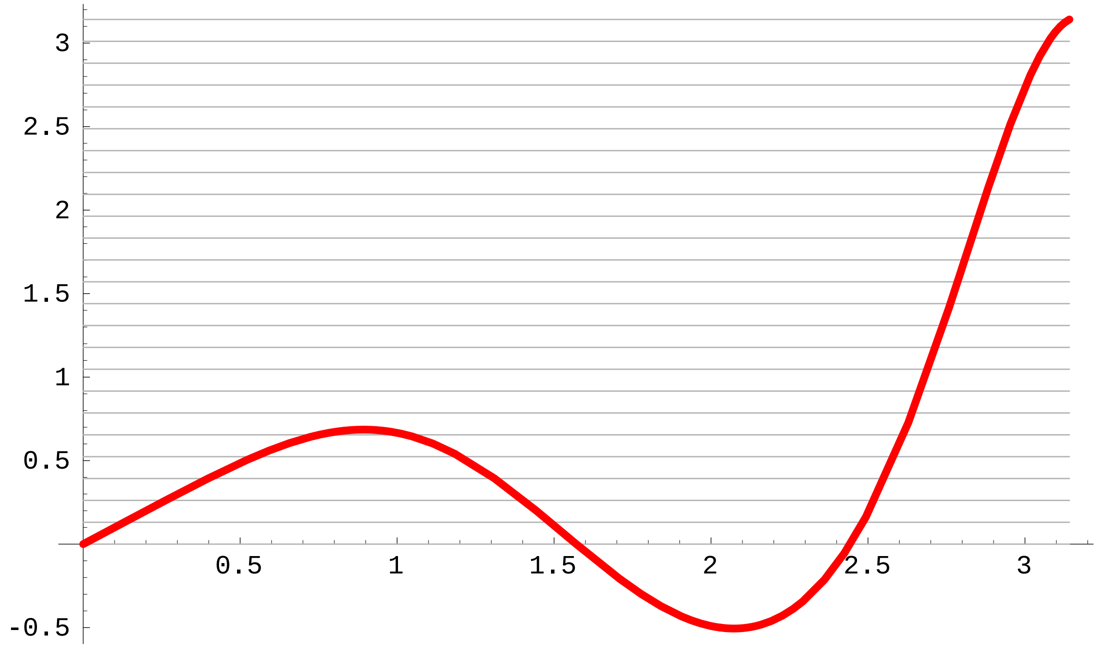
■ 1. Continuous Functions

■ The Intermediate Value Theorem

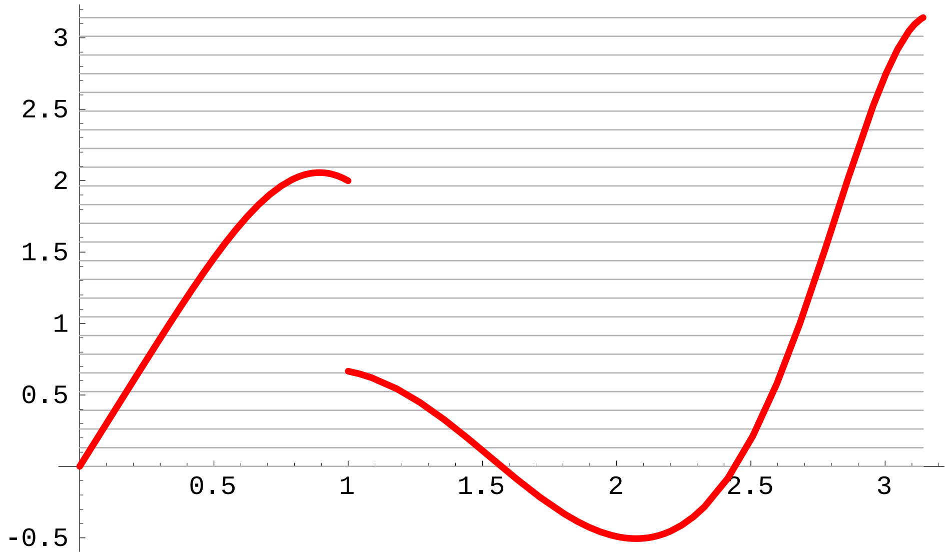
Let f be a continuous function on an interval $[a, b]$. Then for every number c between $f(a)$ and $f(b)$, there is some $x \in [a, b]$ where $f(x) = c$.



Let's say that a function f defined on an interval $[a, b]$ has the **Intermediate Value Property** on $[a, b]$ if for every subinterval $[\alpha, \beta] \in [a, b]$ and every number c between $f(\alpha)$ and $f(\beta)$, there is some $x \in [\alpha, \beta]$ where $f(x) = c$.

The Intermediate Value Theorem says that all continuous functions on $[a, b]$ have the intermediate value property on $[a, b]$.

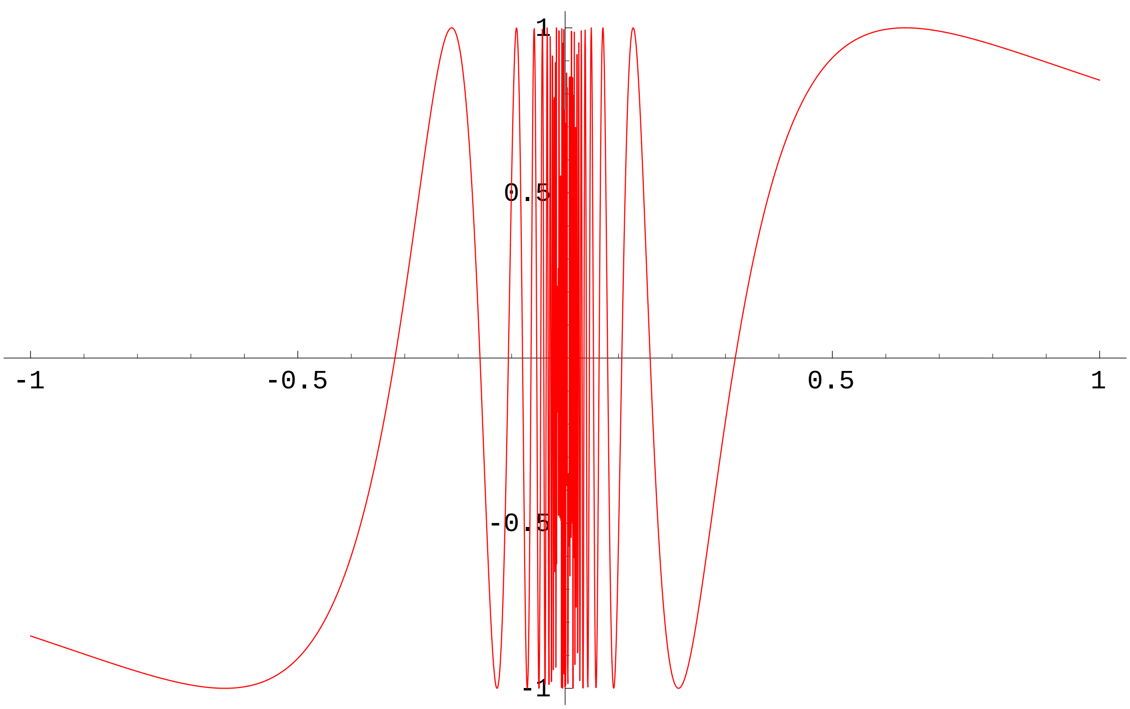
- **Are there *discontinuous* functions on [*a*, *b*] that have the intermediate value property on [*a*, *b*] ?**



Functions with "jump discontinuities" don't. But some functions have another, *nastier* kind of discontinuity.

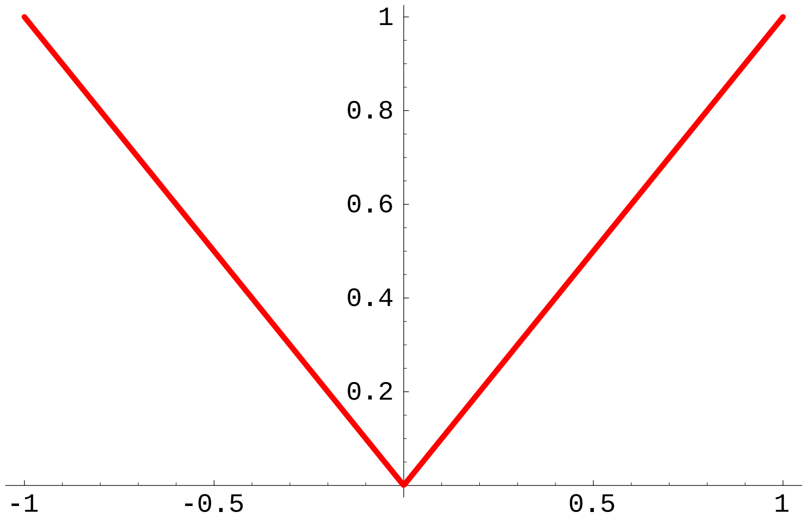
Consider the function

$$f(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \qquad \text{on } [-1, 1].$$

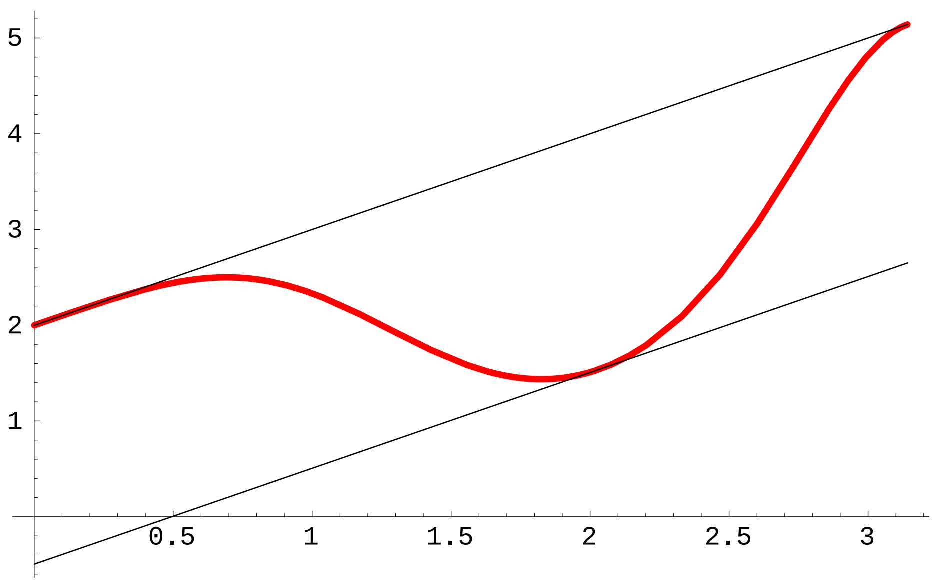


■ 2. Differentiable Functions

All differentiable functions are continuous, but not all continuous functions are differentiable.



■ The Mean Value Theorem



Corollaries

If $f'(x) = 0$ for all $x \in (a, b)$, then $f(x)$ is constant on (a, b).

If $f'(x) = g'(x)$ for all $x \in (a, b)$, then $f(x) - g(x)$ is constant on (a, b).

The Fundamental Theorem of Calculus

If f is continuous on an interval $[a, b]$, then for all $x \in [a, b]$,

$$\frac{d}{dx} \int_a^x f(t)\, dt = f(x).$$

In other words...

Every continuous function f on an interval $[a, b]$ *has* an antiderivative—namely,

$$\int_a^x f(t)\, dt.$$

In still other words...

Every continuous function f on an interval $[a, b]$ is the derivative of some function on $[a, b]$.

But what about all that $F(b) - F(a)$ business?

Combine the theorem above with a corollary to the MVT:

> Any two antiderivatives of a function f on an interval I differ by a constant on I. That is, if $f'(x) = g'(x)$ for all $x \in I$, then $f(x) - g(x)$ is constant on I.

Here's how it goes:

Suppose that f is continuous on $[a, b]$, and that $F(x)$ is any antiderivative of f on $[a, b]$ that you happen to know. Then

$$\int_a^x f(t)\,dt - F(x) = C$$

for all $x \in [a, b]$. The constant C has to equal $-F(a)$. So,

$$\int_a^x f(t)\,dt = F(x) - F(a)$$

for all $x \in [a, b]$, and in particular,

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

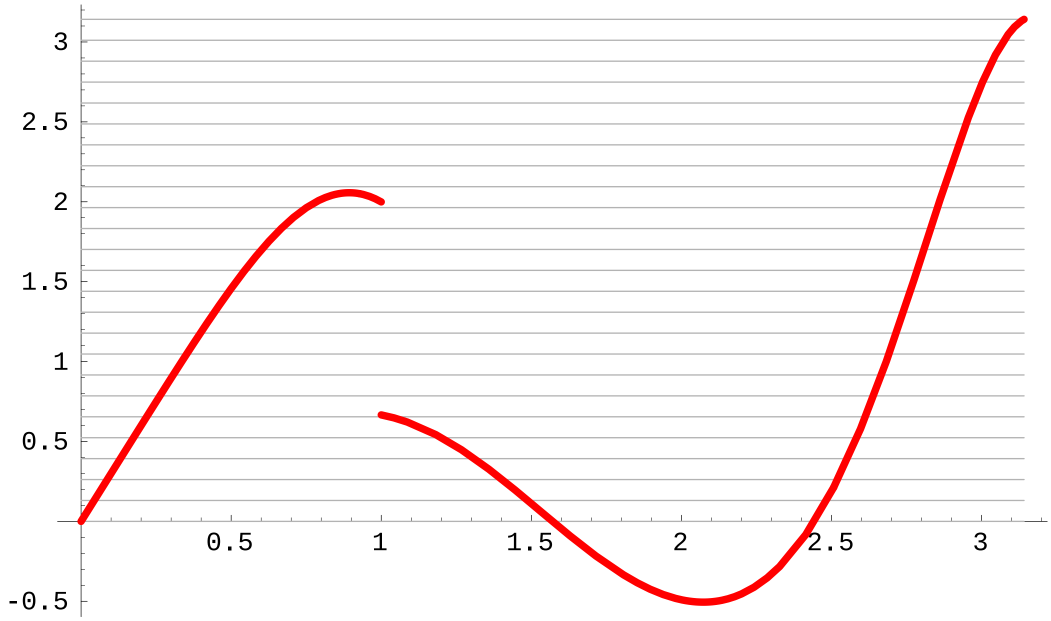
Inquiring minds want to know...

The FTC tells us that every continuous function f on an interval $[a, b]$ is the derivative of some function on $[a, b]$. But...

Are there derivatives that aren't continuous?

That is, are there functions f on $[a, b]$ such that $f'(x)$ exists on $[a, b]$ but f' is not continuous on $[a, b]$?

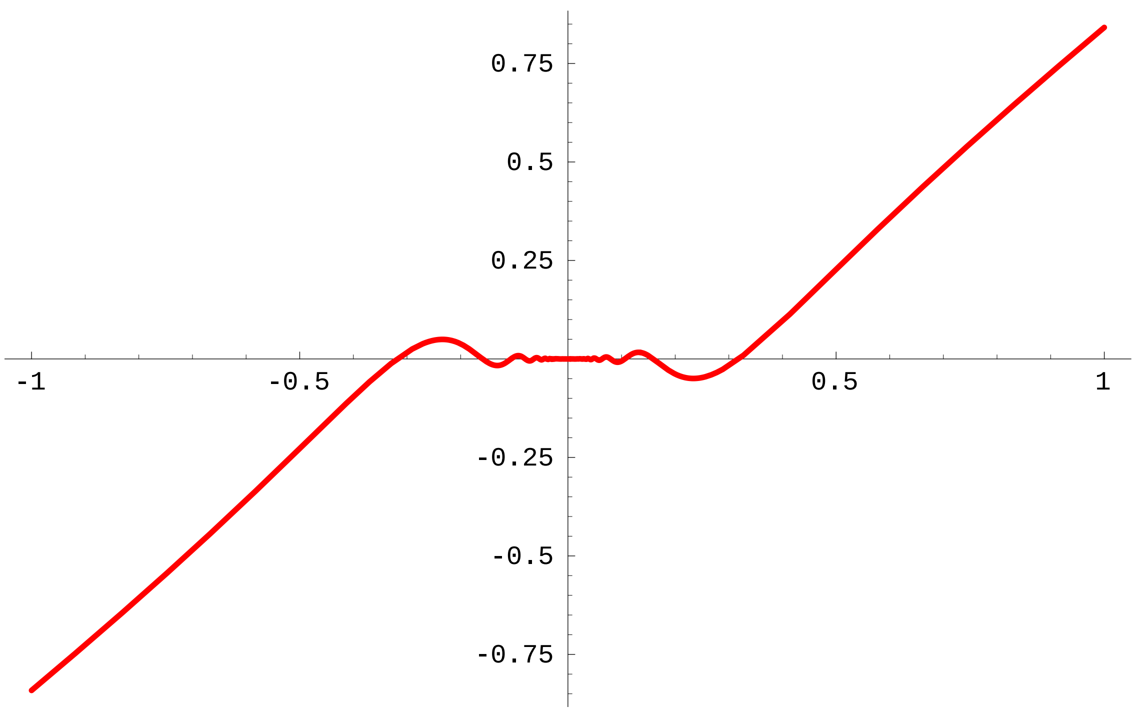
A derivative can't have a jump discontinuity unless it's undefined at the jump. Why?



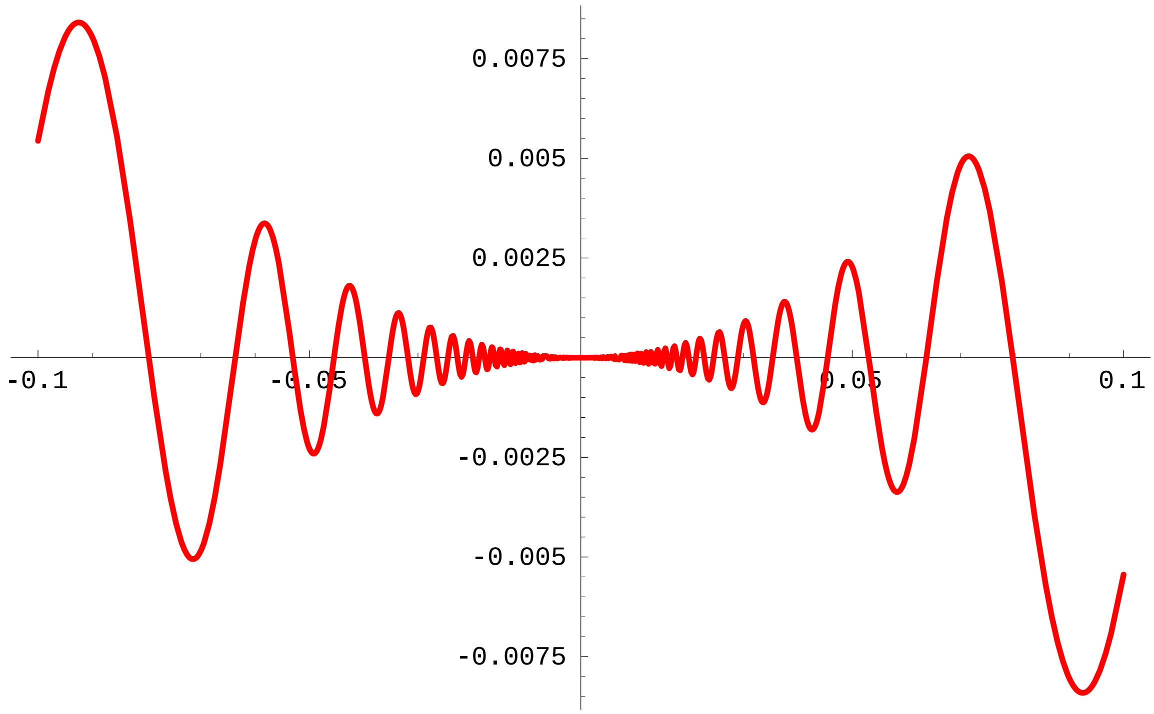
But can a derivative have a "nasty" discontinuity?

Consider this:

$$f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \qquad \text{on } [-1, 1].$$

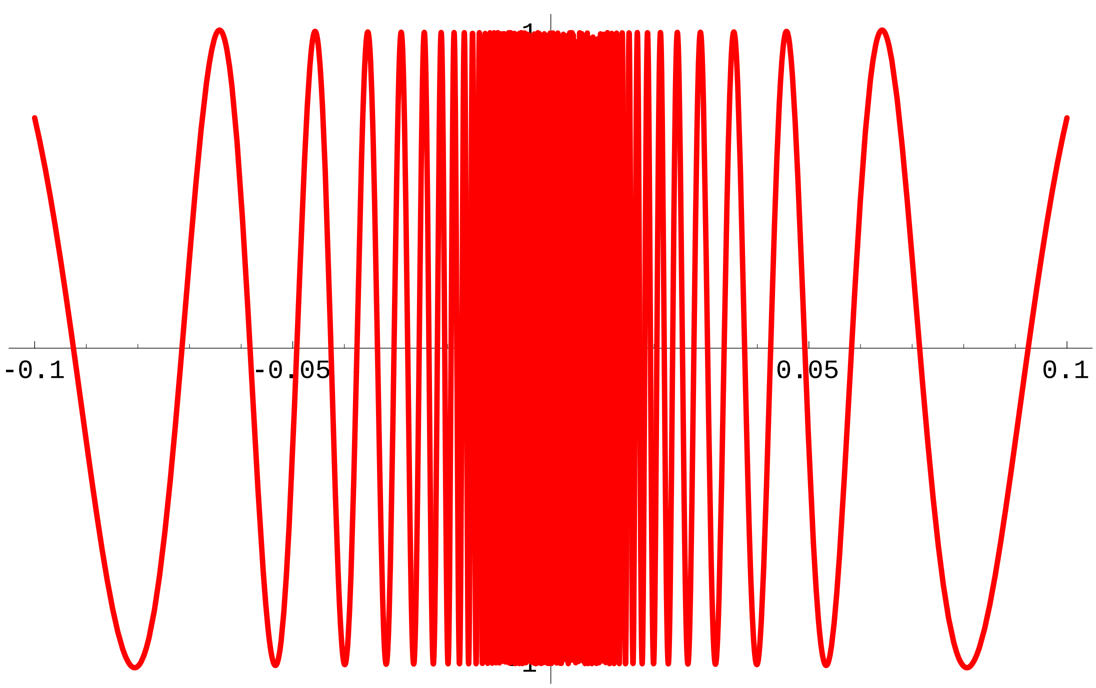


Here's a better view of what's happening near $x = 0$.



And this is the derivative:

$$f'(x) = \begin{cases} 2\,x\sin(1/x) - \cos(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$



So $f'(0)$ exists, while f' is not continuous at $x = 0$.

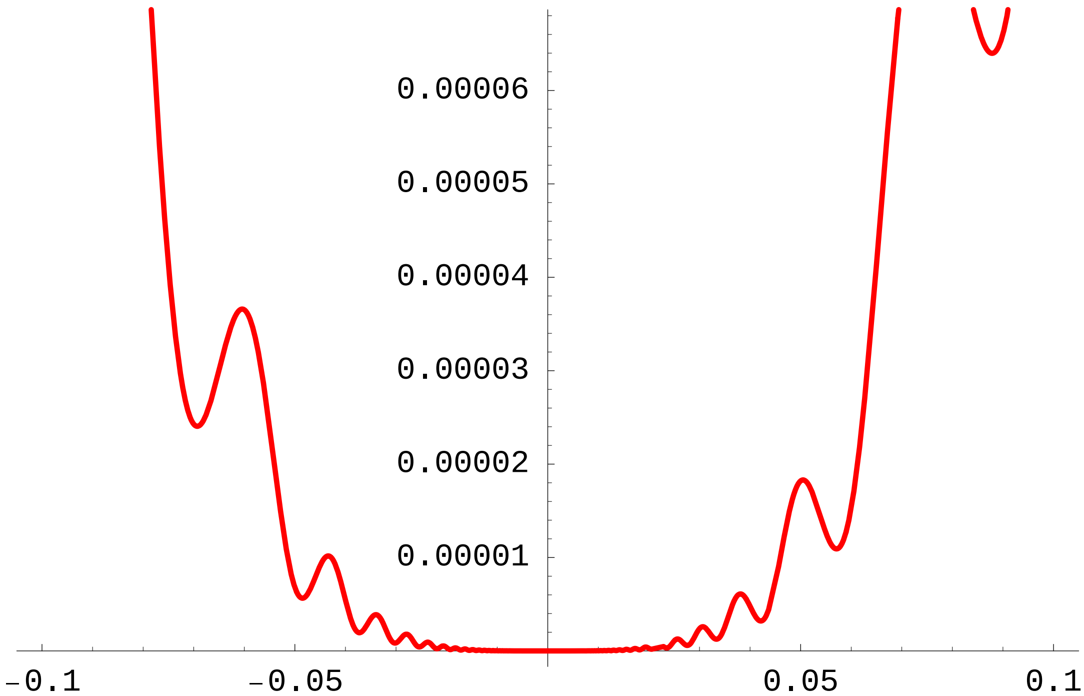
■ A Theorem.

A derivative can be discontinuous at a point where it exists, but it will have the **intermediate value property** on any interval on which it exists.

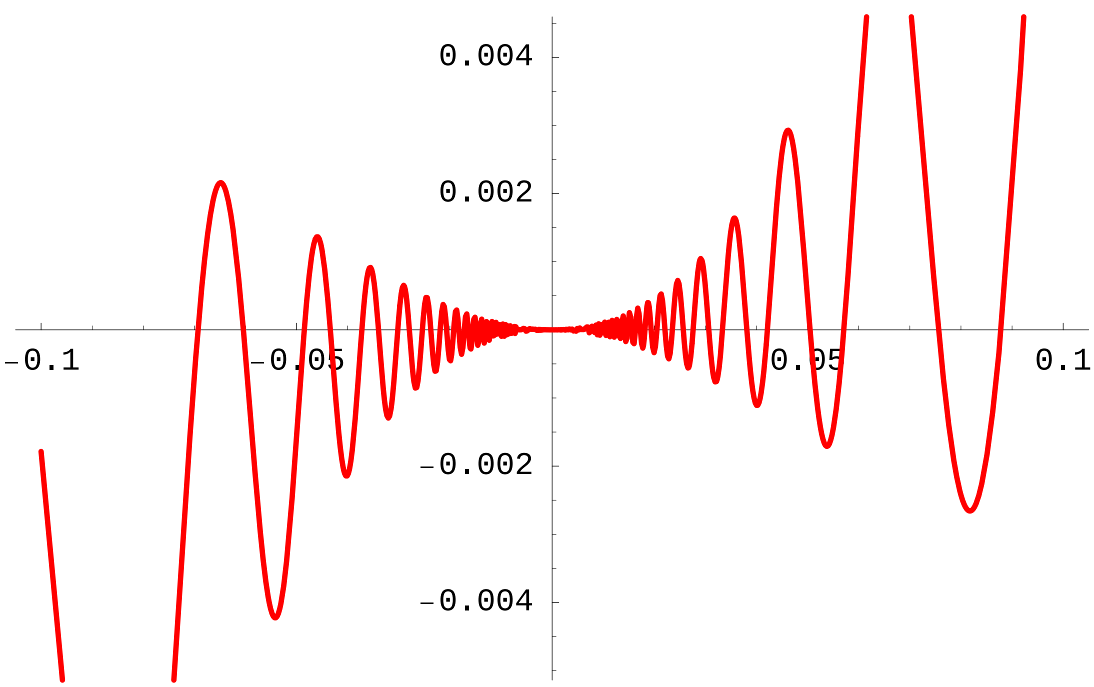
So, while a derivative needn't be a continuous function, it does share one important property with every continuous function.

■ What statement does the following example prove false?

$$f(x) = \begin{cases} x^4\,(2 + \sin(1/x)) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

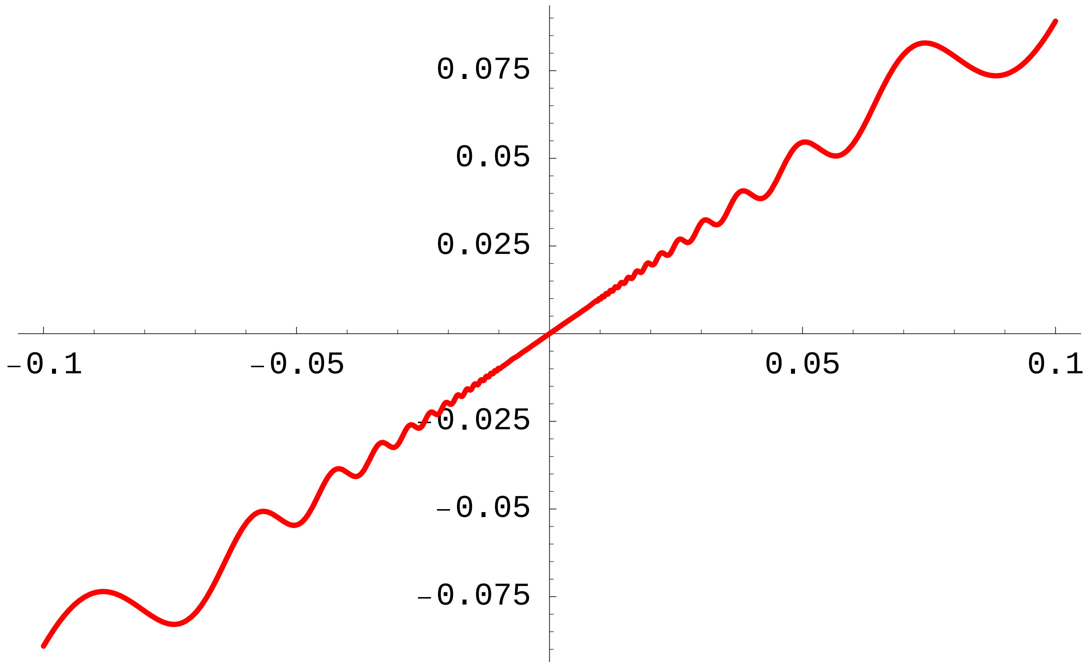


$$f'(x) = \begin{cases} 4\,x^3\,(2 + \sin(1/x)) - x^2\cos(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

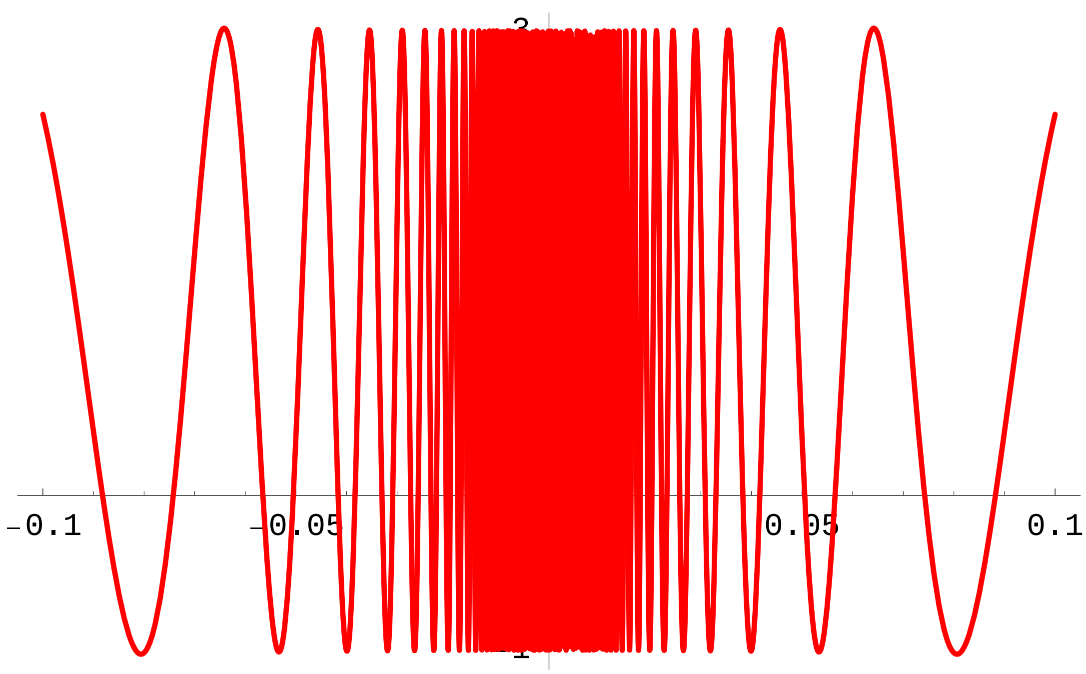


■ What statement does this example prove false?

$$f(x) = \begin{cases} x + 2\,x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \quad \text{on} \quad [-1,\,1].$$



$$f'(x) = \begin{cases} 1 + 4\,x \sin(1/x) - 2\cos(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$



■ The FTC revisited

Every continuous function f on an interval $[a,\ b]$ *has* an antiderivative—namely,

$$\int_a^x f(t)\, dt.$$

Often, such a function has no simpler form. Common examples are known as "**special functions**."

■ Example 1

$$\int_1^x \frac{e^t}{t}\, dt$$

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f[x_] = ∫_1^x ⅇ^t / t ⅆt
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-ExpIntegralEi[1] + ExpIntegralEi[x]
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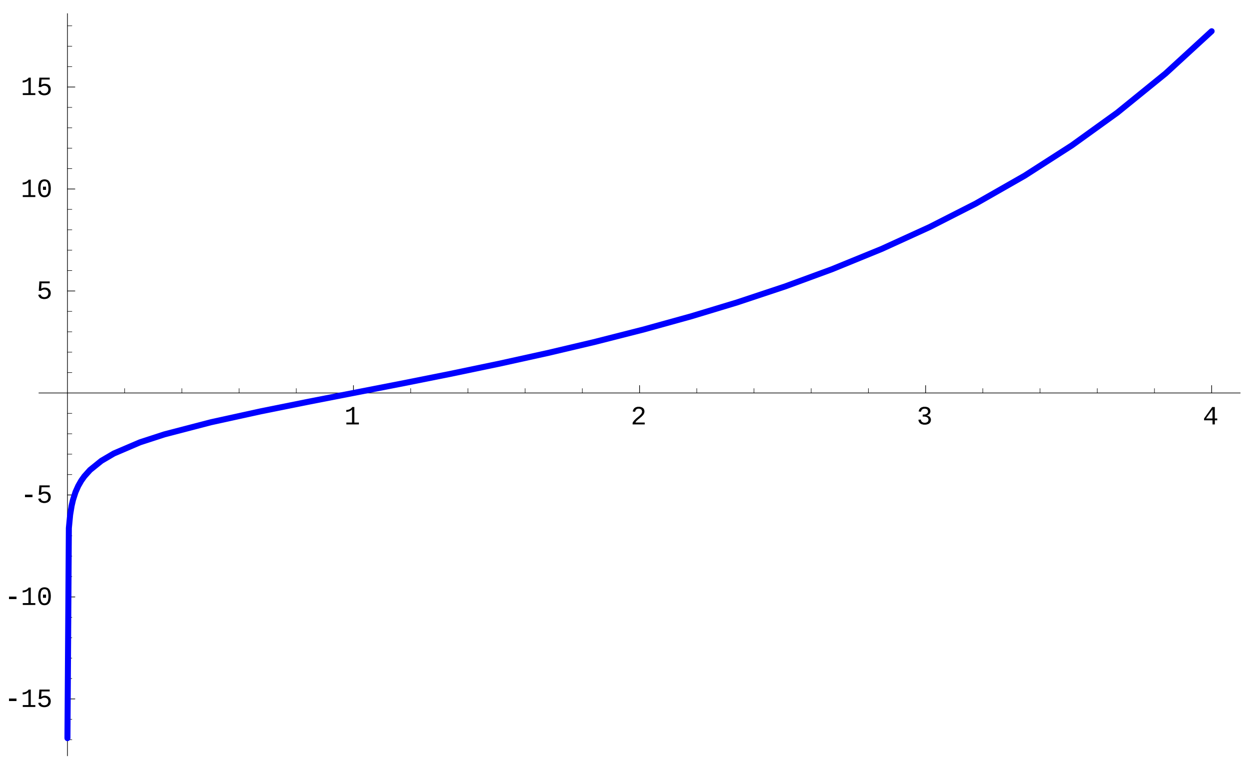
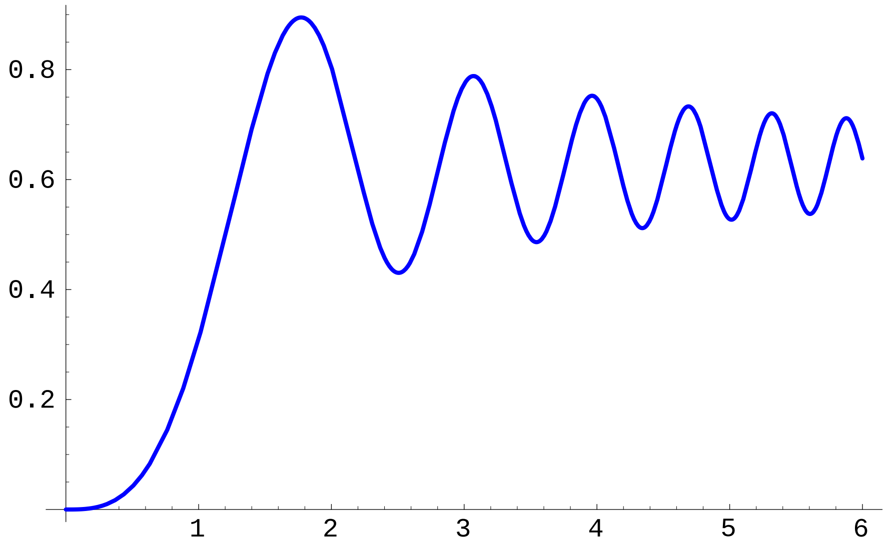


■ Example 2

$$\int_0^x \sin t^2 \, dt$$

$\sqrt{\frac{\pi}{2}}$ FresnelS[$\sqrt{\frac{2}{\pi}}$ x]



■ Example 3

$$\int_0^x e^{-t^2} \, dt$$

$\frac{1}{2}\sqrt{\pi}$ Erf[x]