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THE EXISTENCE OF PERIODIC SOLUTIONS TO REACTION-DIFFUSION SYSTEMS WITH PERIODIC DATA*

J. J. MORGAN† AND S. L. HOLLIS ‡

Abstract. The existence of time-periodic solutions is proven for a large class of reaction-diffusion systems in which Dirichlet boundary data, diffusivities, and reaction rates are periodic with common period.

Key words. periodic solutions, reaction-diffusion systems

AMS subject classifications. 35B10, 35K45, 35K57

1. Introduction.

We consider reaction-diffusion systems of the form

$$
(1.1)\qquad
\begin{aligned}
\frac{\partial u_i}{\partial t} - d_i(t)\Delta u_i &= f_i(x,t,u) && \text{in } \Omega\times\{t>0\},\ i=1,\ldots,m\\
u_i(x,t) &= g_i(x,t) && \text{on } \partial\Omega\times\{t>0\},\ i=1,\ldots,m\\
u_i(x,0) &= u_{0_i}(x) && \text{on } \overline{\Omega},\ i=1,\ldots,m
\end{aligned}
$$

where $u = (u_i)_{i=1}^m$ and Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$; *i.e.,* $\partial\Omega$ is an $n-1$ dimensional $C^{2+\alpha}$ manifold of which Ω lies locally on one side. We assume that the initial data u_{0_i} are bounded, measurable, and nonnegative and each $d_i \in C(\mathbb{R}_+;[a,b])$ where $0 < a \le b < \infty$. (The symbol $\mathbb{R}_+$ denotes $[0,\infty)$.) We also assume that the reaction functions f_i are continuous on $\overline{\Omega}\times\mathbb{R}_+\times\mathbb{R}_+^m$ and locally Lipschitz in u and that $f = (f_i)_{i=1}^m$ is *quasi-positive*; *i.e.,* for each $i = 1,\ldots,m$, we have $f_i(\,\cdot\,,\cdot\,,\xi) \ge 0$ for all $\xi \ge 0$ with $\xi_i = 0$. Each g_i is assumed to be a nonnegative member of $C^{2,1}(\partial\Omega\times\mathbb{R}_+)$. These standard basic assumptions guarantee *local* existence of unique, nonnegative, classical solutions on a maximal time interval $0 \le t < T^* \le \infty$. This follows from straightforward adaptation of results in, *e.g.,* [5, 13] to account for the t-dependence of the parameters in (1.1).

In addition to the basic assumptions stated above, we assume the following.

(A1) There is a $K > 0$ and for each $i = 1,\ldots,m$ there are nonnegative constants $\alpha_{i,1},\alpha_{i,2},\ldots,\alpha_{i,i}$ with $\alpha_{i,i} > 0$ such that

$$\sum_{j=1}^{i}\alpha_{i,j}f_j(x,t,\xi) \;\le\; K\Big(1+\sum_{j=1}^{m}\xi_j\Big) \quad \text{for all } x\in\Omega,\ t\ge 0 \text{ and } \xi\in\mathbb{R}_+^m.$$

(A2) Each $|f_i(\,\cdot\,,\cdot\,,\xi)|$, $i=1,\ldots,m$, is bounded above by a polynomial in $\xi_1,\ldots,\xi_m$.

The following global existence theorem follows from results in Morgan [10].

THEOREM 1.1. *Let the conditions (A1) and (A2) be met. Then for any bounded, measurable, nonnegative initial data $u_0 = (u_{0_i})_{i=1}^m$, we have $T^* = \infty$; that is, the system (1.1) has a unique, nonnegative, classical solution on $\overline{\Omega}\times[0,\infty)$.*

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†Department of Mathematics, Texas A&M University, College Station, TX 77843. The research of this author was supported in part by NSF grant DMS-9208046.

‡Department of Mathematics and Computer Science, Armstrong State College, Savannah, GA 31419.

Remark. We refer to (A1) as a linear "intermediate sums" condition [10]. It allows high order nonlinearities in the individual f_i but requires that f_1 be bounded above by a linear polynomial in u and cancellation of high order positive terms in the intermediate sums. An illustrative example is the following three-species predator-prey system.

$$\begin{aligned}\frac{\partial u_1}{\partial t} - d_1\Delta u_1 &= k_{11}u_1(M-u_1) - k_{12}u_1u_2 - k_{13}u_1u_3\\ \frac{\partial u_2}{\partial t} - d_2\Delta u_2 &= k_{12}u_1u_2 - k_{23}u_2u_3 - k_{20}u_2\\ \frac{\partial u_3}{\partial t} - d_3\Delta u_3 &= k_{13}u_1u_3 + k_{23}u_2u_3 - k_{30}u_3\end{aligned} \tag{1.2}$$

where M and the k_{ij} are bounded, nonnegative, continuous functions on $\overline{\Omega}\times\mathbb{R}_+$. Here one can take $\alpha_{2,1} = \alpha_{2,2} = 1$ and $\alpha_{3,1} = \alpha_{3,2} = \alpha_{3,3} = 1$. Note that in this example we actually have $\sum_{j=1}^{i} \alpha_{i,j} f_j(x,t,u) \le K$ due to the assumption of logistic growth of u_1 when $u_2 = u_3 = 0$. Exponential growth of one or more of the species in the absence of the others would lead to a condition of precisely the type in (A1).

An intermediate sums condition of the form (A1) is indeed satisfied by a variety of complex models of, *e.g.,* population dynamics, chemical reactions, and spread of disease [10, 11, 12]. We remark also that nonlinear intermediate sums are possible with the allowable order depending upon the spatial dimension n; see [10].

Our concern here is the existence of a time-periodic solution to (1.1) in the situation where the reaction function f, the diffusivities d_i, and the boundary data g are each periodic in t with common period T. Of particular interest from the point of view of applications would be, for example,

- population models and models of the spread of disease in which birth and death rates, rates of diffusion, rates of infection/interaction, and environmental carrying capacities are periodic on a seasonal scale;
- chemical reaction models in which reaction rates and diffusivities are periodic on a daily scale due to oscillations in sunlight and/or temperature.

With this issue in mind, we assume the following.

(A3) There is a $T > 0$ such that for $i = 1,\dots,m$ and $t \ge 0$ we have

$$f_i(\,\cdot\,,t,\,\cdot\,) = f_i(\,\cdot\,,t+T,\,\cdot\,),\quad g_i(\,\cdot\,,t) = g_i(\,\cdot\,,t+T),\quad \text{and}\quad d_i(t) = d_i(t+T).$$

(A4) There is a continuous function $\tilde{g} : \overline{\Omega} \to \mathbb{R}^m_+$ such that $g(\,\cdot\,,0) = g(\,\cdot\,,T) = \tilde{g}|_{\partial\Omega}$.

(A5) The constants K and $\alpha_{m,1},\dots,\alpha_{m,m}$ in (A1) may be chosen so that $\alpha_{m,j} > 0$ for $j = 1,\dots,m$ and so that

$$\sum_{j=1}^{m} \alpha_{m,j} f_j(x,t,\xi) \;\le\; K \quad \text{for all } x\in\Omega,\ t\ge 0 \text{ and } \xi\in\mathbb{R}^m_+.$$

Note that (A5) is satisfied by the example system (1.2). This would also be true of more general population models of this type provided that each species exhibits bounded growth in the absence of all other species.

Our main result is the following theorem.

THEOREM 1.2. *Under assumptions (A1)–(A5), there exists a* $u_0 \in C(\overline{\Omega};\mathbb{R}^m_+)$ *such that the solution of (1.1) satisfies* $u(\,\cdot\,,t) = u(\,\cdot\,,t+T)$ *on* $\overline{\Omega}$ *for all* $t \ge 0$.

Previous results along these lines can be found in Liu and Pao [9], where the existence of a (unique) T-periodic solution is established via the contraction mapping theorem in the case of a one-dimensional domain and under somewhat stringent conditions on the diffusion coefficients and reaction rates. Our approach will use a variation on Schauder's theorem and will require no assumptions other than (A1)–(A5) to establish the existence of a T-periodic solution. Related work in which scalar parabolic equations are considered includes [1, 2, 3, 14].

2. Formulation of the fixed point problem. We will use the following corollary to Schauder's theorem. For the proof, see, *e.g.,* Gilbarg and Trudinger [4, Thm. 11.3].

THEOREM 2.1. *Let X be a Banach space and $F : X \to X$ a compact map. Assume that there exists a constant $C > 0$ such that $\|z\| < C$ for all z satisfying $z = \sigma F z$ with $\sigma \in (0, 1)$. Then there exists a fixed point z^* of F satisfying $\|z^*\| \leq C$.*

For convenience of notation, let us define the formal solution operator for (1.1) by $\mathcal{S}(t)u_0 = u(\,\cdot\,, t)$ for $t \geq 0$. Now define $F : C_0(\overline{\Omega}; \mathbb{R}^m) \to C_0(\overline{\Omega}; \mathbb{R}^m)$ by

$$Fz = \mathcal{S}(T)(z + \tilde{g})^+ - \tilde{g}, \tag{2.1}$$

where

$$C_0(\overline{\Omega}; \mathbb{R}^m) = \Big\{ z \in C(\overline{\Omega}; \mathbb{R}^m) \;\Big|\; z = 0 \text{ on } \partial\Omega \Big\}$$

and T and $\tilde{g}$ are as in (A3) and (A4). By parabolic regularity [8], F is a compact map. Note also that if z^* is a fixed point of F, then $z^* + \tilde{g} = \mathcal{S}(T)(z^* + \tilde{g})^+$. Consequently $z^* + \tilde{g} \geq 0$, and so $u_0^* \equiv z^* + \tilde{g}$ is a (nonnegative) fixed point of $\mathcal{S}(T)$. So the existence of a T-perodic solution of (1.1) will follow from the existence of a fixed point of the operator F because of (A3) and uniqueness of solutions to (1.1).

Suppose that $0 < \sigma < 1$ and that $z = \sigma F z$. Also set $u_0 = z + \tilde{g}$. Then we see that

$$u_0 = \sigma \mathcal{S}(T) u_0^+ + (1 - \sigma)\tilde{g}.$$

But $z = \sigma F z$ implies that $z + \sigma\tilde{g} \geq 0$, which then implies that $u_0 \geq 0$. Thus

$$u_0 = \sigma \mathcal{S}(T) u_0 + (1 - \sigma)\tilde{g}. \tag{2.2}$$

Let us now define the set

$$\Lambda_T = \Big\{ u_0 \in C(\overline{\Omega}; \mathbb{R}^m_+) \;\Big|\; u_0 = \sigma \mathcal{S}(T) u_0 + (1 - \sigma)\tilde{g} \;\text{ for some } \sigma \in (0, 1) \Big\}. \tag{2.3}$$

In light of Theorem 2.1 and the preceding observations, our goal is to show that Λ_T is a bounded subset of $C(\overline{\Omega}; \mathbb{R}^m_+)$. Note that because of (2.2) this can be accomplished by showing that there is a $C > 0$ such that $\|\mathcal{S}(T)u_0\|_\infty \leq C$ for all $u_0 \in \Lambda_T$.

3. A preliminary estimate. Our first step toward showing that Λ_T is a bounded subset of $C(\overline{\Omega}; \mathbb{R}^m_+)$ is the following L^1 estimate.

LEMMA 3.1. *Suppose that (A1)–(A5) are true. Then there is a constant $C_1 > 0$ such that*

$$\|u_i\|_{1,\Omega\times(0,T)} \leq C_1, \quad i = 1, \ldots, m$$

for all u satisfying (1.1) with $u_0 \in \Lambda_T$.

Proof. Let $u_0 \in \Lambda_T$ and let u be the corresponding solution of (1.1). Also define $w \equiv \int_0^T \sum_{k=1}^m \alpha_{m,k} d_k(s) u_k(\,\cdot\,, s) ds$. Summing the equations in (1.1), applying (A5), and integrating over $t \in [0, T]$ yields

$$\sum_{k=1}^{m} \alpha_{m,k}\big(u_k(\,\cdot\,,T) - u_{0_k}\big) - \Delta w \;\le\; KT \quad \text{on } \Omega. \tag{3.1}$$

For convenience, set $v = F(u_0 - \tilde{g})$ where F is as in (2.1). Then $u(\,\cdot\,,T) = v + \tilde{g}$ and $u_0 = \sigma v + \tilde{g}$. So (3.1) becomes

$$\sum_{k=1}^{m} \alpha_{m,k}(v_k + \tilde{g}_k) - \Delta w \;\le\; \sum_{k=1}^{m} \alpha_{m,k}(\sigma v_k + \tilde{g}_k) + KT \quad \text{on } \Omega.$$

Hence

$$-\Delta w \;\le\; (\sigma - 1)\sum_{k=1}^{m} \alpha_{m,k} v_k + KT \;\le\; (1-\sigma)\sum_{k=1}^{m} \alpha_{m,k}\tilde{g}_k + KT \quad \text{on } \Omega$$

since $v_k \ge -\tilde{g}_k$. Also, on $\partial\Omega$ we have $w = \int_0^T \sum_{k=1}^m \alpha_{m,k} d_k(s) g_k(\,\cdot\,, s) ds$. Therefore, one can apply a comparison principle and nonnegativity to obtain a bound on $\|w\|_{\infty,\Omega}$ and in turn a bound on $\|\sum_{k=1}^m u_k\|_{1,\Omega\times(0,T)}$, where each bound is independent of u_0 and σ. ∎

Remark. This result remains true without (A1) and (A2), provided that the interval $[0, T]$ lies within the maximal interval of existence $[0, T^*)$.

4. The bootstrapping framework. The following lemma provides a bootstrapping mechanism for obtaining L^p estimates for large p from an L^1 estimate. Although the proof is essentially the same as that of similar results in [6, 7, 10, 11], we include it here for the sake of completeness.

LEMMA 4.1. *Suppose that (A1) is true and u satisfies (1.1) for $0 \le t < T$. Let $\tau \in [0, T)$. There is a constant C, independent of u and τ, such that the following are valid for $k = 1, \ldots, m$.*

i) If $1 < p < \frac{n+2}{n}$, then

$$\|u_k\|_{p,\Omega\times(\tau,T)} \le C\Big[\,1 + \Big\|\sum_{i=1}^{m} u_i(\,\cdot\,,\tau)\Big\|_{1,\Omega} + \Big\|\sum_{i=1}^{m} u_i\Big\|_{1,\Omega\times(\tau,T)}\Big].$$

ii) If $p > \frac{n+2}{n}$ and $r > \frac{np}{n+2}$, then

$$\|u_k\|_{p,\Omega\times(\tau,T)} \le C\Big[\,1 + \Big\|\sum_{i=1}^{m} u_i(\,\cdot\,,\tau)\Big\|_{r,\Omega} + \Big\|\sum_{i=1}^{m} u_i\Big\|_{r,\Omega\times(\tau,T)}\Big].$$

A central role in the proof of Lemma 4.1 is played by the solution of the scalar equation

$$\begin{aligned} \frac{\partial \chi}{\partial t} - d\,\Delta\chi &= \vartheta && \text{in } \Omega\times(\tau,T) \\ \chi &= 0 && \text{on } \partial\Omega\times(\tau,T) \\ \chi(\,\cdot\,,\tau) &= 0 && \text{on } \Omega \end{aligned} \tag{4.1}$$

where $\tau < T$ and $d \in C(\mathbb{R}_+; [a,b])$ with $0 < a \le b < \infty$. We now state some more or less well-known L^q regularity results for (4.1).

LEMMA 4.2. *Let $1 < q < \infty$ and suppose that $\vartheta \in L^q(\Omega\times(\tau,T);\mathbb{R}_+)$ where $0 \le \tau < T$. Then (4.1) has a unique solution $\chi \in W_q^{2,1}(\Omega\times(\tau,T);\mathbb{R}_+)$. If $\|\vartheta\|_{q,\Omega\times(\tau,T)} = 1$, then there exists a constant $C = C(q,T)$, independent of ϑ and τ, such that $\|\chi\|_{W_q^{2,1}(\Omega\times(\tau,T))} \le C$. Furthermore, C can be chosen so that:*

(i) $\|\chi(\,\cdot\,,T)\|_{q,\Omega} \le C$;

(ii) if $q > \frac{n+2}{2}$, *then* $\|\chi\|_{\infty,\Omega\times(\tau,T)} \le C$;

(iii) if $1 < q < \frac{n+2}{2}$ *and* $1 < s < \frac{nq}{n-2(q-1)}$, *then*

$$\|\chi\|_{s,\Omega\times(\tau,T)} \le C \quad \text{and} \quad \|\chi(\,\cdot\,,T)\|_{s,\Omega} \le C;$$

(iv) $\|\chi\|_{W_q^{1,0}(\partial\Omega\times(0,T))} \le C$.

For proof of these results, we refer to sections IV.9 and II.3 of Ladyženskaja, *et al.* [5] and section 3 of Morgan [11]. We now proceed with the

Proof of Lemma 4.1. Let $p > 1$ and $q = \frac{p}{p-1}$, and let $\vartheta \in L^q(\Omega\times(\tau,T);\mathbb{R}_+)$ with $\|\vartheta\|_{q,\Omega\times(\tau,T)} = 1$. Take $k \in \{1,2,\ldots,m\}$ and let χ be the solution of (4.1) with $d = d_k$. Now for $t \in (\tau,T]$ define $\varphi(\,\cdot\,,t) = \chi(\,\cdot\,,T+\tau-t)$ and $\overline{\vartheta}(\,\cdot\,,t) = \vartheta(\,\cdot\,,T+\tau-t)$ so that φ satisfies

$$\begin{aligned} \frac{\partial\varphi}{\partial t} + d_k\,\Delta\varphi &= -\overline{\vartheta} && \text{in } \Omega\times(\tau,T) \\ \varphi &= 0 && \text{on } \partial\Omega\times(\tau,T) \\ \varphi(\,\cdot\,,T) &= 0 && \text{on } \Omega. \end{aligned}$$

We integrate $\overline{\vartheta}\sum_{i=1}^k \alpha_{k,i}u_i$ over $\Omega\times(\tau,T)$ and obtain

$$\begin{aligned} \int_\tau^T\!\!\int_\Omega \overline{\vartheta}\sum_{i=1}^k \alpha_{k,i}u_i \;\le\; &\int_\Omega \varphi(\,\cdot\,,\tau)\sum_{i=1}^k \alpha_{k,i}u_i(\,\cdot\,,\tau) + K\int_\tau^T\!\!\int_\Omega\Big(1+\sum_{i=1}^m u_i\Big)\varphi \\ &+ \int_\tau^T\!\!\int_\Omega \Delta\varphi\sum_{i=1}^k \alpha_{k,i}(d_k-d_i)u_i - \int_\tau^T\!\!\int_{\partial\Omega}\frac{\partial\varphi}{\partial \mathrm{n}}\sum_{i=1}^k \alpha_{k,i}d_i g_i. \end{aligned}$$

Now with $1 \le r \le \infty$ and $s = \frac{r}{r-1}$, Hölder's inequality gives

$$\begin{aligned} \int_\tau^T\!\!\int_\Omega \overline{\vartheta}\sum_{i=1}^k \alpha_{k,i}u_i \;\le C\Bigg(&\Big\|\sum_{i=1}^k u_i(\,\cdot\,,\tau)\Big\|_{r,\Omega}\|\varphi(\,\cdot\,,\tau)\|_{s,\Omega} \\ &+\Big\|1+\sum_{i=1}^m u_i\Big\|_{r,\Omega\times(\tau,T)}\|\varphi\|_{s,\Omega\times(\tau,T)} \\ &+\Big\|\sum_{i=1}^{k-1} u_i\Big\|_{p,\Omega\times(\tau,T)}\|\Delta\varphi\|_{q,\Omega\times(\tau,T)} + \Big\|\frac{\partial\varphi}{\partial\mathrm{n}}\Big\|_{q,\partial\Omega\times(\tau,T)}\Bigg) \end{aligned} \tag{4.2}$$

for some $C > 0$. If $1 < p < \frac{n+2}{n}$, then $q > \frac{n+2}{2}$, and so by Lemma 4.2 we can take $r = 1$ and $s = \infty$ and obtain by duality that

$$\|u_k\|_{p,\Omega\times(\tau,T)} \le C_p\Big(1 + \Big\|\sum_{i=1}^k u_i(\,\cdot\,,\tau)\Big\|_{1,\Omega} + \Big\|\sum_{i=1}^m u_i\Big\|_{1,\Omega\times(\tau,T)} + \Big\|\sum_{i=1}^{k-1} u_i\Big\|_{p,\Omega\times(\tau,T)}\Big)$$

for some $C_p > 0$. From this follows part i) of the lemma by induction on k. Now suppose that $p > \frac{n+2}{n}$ and $r > \frac{np}{n+2}$. Then we have $q < \frac{n+2}{2}$ and $s < \frac{np}{np-(n+2)} = \frac{nq}{n-2(q-1)}$. So from (4.2) and Lemma 4.2 we have by duality that

$$\|u_k\|_{p,\Omega\times(\tau,T)} \le C_p\Big(1 + \Big\|\sum_{i=1}^{k} u_i(\,\cdot\,,\tau)\Big\|_{r,\Omega} + \Big\|\sum_{i=1}^{m} u_i\Big\|_{r,\Omega\times(\tau,T)} + \Big\|\sum_{i=1}^{k-1} u_i\Big\|_{p,\Omega\times(\tau,T)}\Big)$$

for some $C_p > 0$. Part ii) of the lemma now follows by induction on k. □

5. The proof of Theorem 1.2.

We begin this section with one more lemma.

LEMMA 5.1. *Assume (A1)–(A5). There exist sequences* $\{C_k\}_{k=1}^\infty \subset (0,\infty)$ *and* $\{p_k\}_{k=1}^\infty \subset [1,\infty)$ *with* $p_k \uparrow \infty$ *such that if* u *satisfies (1.1) with* $u_0 \in \Lambda_T$ *then for* $i = 1,\dots,m$ *and* $k \in \mathbb{N}$ *we have* $\|u_i\|_{p_k,\Omega\times(t_k,T)} \le C_k$ *where* $t_k = (1-2^{1-k})T$.

Proof. First we take $p_1 = 1$ and use the C_1 from Lemma 3.1. By that same L^1 estimate, there is a $\tau_1 \in (0, \frac{1}{2}T)$ such that

$$\|u_i(\,\cdot\,,\tau_1)\|_{1,\Omega} \le \frac{C_1}{T/2}, \quad i = 1,\dots,m.$$

Now set $p_2 = \left(\frac{n+2}{n}\right)^{3/4}$. By part i) of Lemma 4.1 there exists a C_2 such that

$$\|u_i\|_{p_2,\Omega\times(\tau_1,T)} \le C_2, \quad i = 1,\dots,m,$$

and thus $\|u_i\|_{p_2,\Omega\times(\frac{1}{2}T,T)} \le C_2,\ i = 1,\dots,m$. Therefore there is a $\tau_2 \in (\frac{1}{2}T, \frac{3}{4}T)$ such that

$$\|u_i(\,\cdot\,,\tau_2)\|_{p_2,\Omega} \le \frac{C_2}{(T/4)^{1/p_2}}, \quad i = 1,\dots,m.$$

Now in part ii) of Lemma 4.1 we take $r = p_2$ and $p = p_3 \equiv \left(\frac{n+2}{n}\right)^{3/2}$ and obtain a C_3 so that $\|u_i\|_{p_3,\Omega\times(\tau_2,T)} \le C_3,\ i = 1,\dots,m$, and consequently $\|u_i\|_{p_3,\Omega\times(\frac{3}{4}T,T)} \le C_3,\ i = 1,\dots,m$. Now we can choose $\tau_3 \in (\frac{3}{4}T, \frac{7}{8}T)$ such that

$$\|u_i(\,\cdot\,,\tau_3)\|_{p_3,\Omega} \le \frac{C_3}{(T/8)^{1/p_3}}, \quad i = 1,\dots,m,$$

and obtain similarly a C_4 such that $\|u_i\|_{p_4,\Omega\times(\frac{7}{8}T,T)} \le C_4,\ i = 1,\dots,m$, where $p_4 \equiv \left(\frac{n+2}{n}\right)^2$. Continuing in this way, we take $p_k = \left(\frac{n+2}{n}\right)^{k/2}$ for $k = 5,6,\dots$ and obtain corresponding C_k such that $\|u_i\|_{p_k,\Omega\times((1-2^{1-k})T,T)} \le C_k,\ i = 1,\dots,m$. □

The preceding lemma gives rise to the following key result.

COROLLARY 5.2. *Assume (A1)–(A5). There exist* $C^* > 0$ *and* $t^* \in (0,T)$ *such that*

$$\|u_i\|_{\infty,\Omega\times(t^*,T)} \le C^*, \quad i = 1,\dots,m$$

for all u *satisfying (1.1) with* $u_0 \in \Lambda_T$.

Proof. Suppose that u satisfies (1.1) with $u_0 \in \Lambda_T$. By the polynomial growth assumption (A5), we can choose k sufficiently large in Lemma 5.1 so that each $f_i(\,\cdot\,,\cdot\,,u)$ is in $L^{\frac{n+2}{2}}\big(\Omega \times ((1-2^{1-k})T,\, T)\big)$ and at the same time each u_i is in

$L^2\big(\Omega \times ((1-2^{1-k})T,\, T)\big)$ with each norm bounded independent of u. Consequently, we can apply Theorem III.8.1 of Ladyženskaja, *et al.* [8], to obtain the desired result where $t^* = (1-2^{-k})T$. ∎

We are now ready to complete the
Proof of Theorem 1.3. From Corollary 5.2 it follows that

$$\|u_i(\,\cdot\,,T)\|_{\infty,\Omega} \le C^*, \quad i = 1,\dots,m,$$

for all u satisfying (1.1) with $u_0 \in \Lambda_T$. Thus by (2.2) there is a constant $\widetilde{C}$ such that

$$\|u_{0_i}\|_{\infty,\Omega} \le \widetilde{C}, \quad i = 1,\dots,m,$$

for all $u_0 \in \Lambda_T$. That is, Λ_T is a bounded subset of $C(\overline{\Omega};\mathbb{R}^m_+)$. So by Theorem 2.1 and the discussion in section 2, the mapping F defined by (2.1) has a fixed point $z^* \in C_0(\overline{\Omega};\mathbb{R}^m)$, which gives rise to a fixed point $u_0^* = z^* + \tilde{g} \in C(\overline{\Omega};\mathbb{R}^m_+)$ of $\mathcal{S}(T)$. Now by the periodicity of f and g and the uniqueness of solutions to (1.1), it follows that (1.1) possesses the T-periodic solution $u(\,\cdot\,,t) = \mathcal{S}(t)u_0^*$. ∎

6. Remarks and generalizations. Straightforward modifications of our proofs show that Theorem 1.2 remains valid if the boundary conditions are of Robin type with smooth, T-periodic parameters. If the boundary conditions are Neumann type, then (A5) must be modified so that

$$\sum_{j=1}^{m} \alpha_{m,j} f_j(\,\cdot\,,\,\cdot\,,u) \le K - \epsilon \sum_{j=1}^{m} u_j \tag{6.1}$$

for some $\epsilon > 0$ in order to obtain the L^1 estimate in Lemma 3.1. Also, if boundary conditions are Dirichlet or Robin type, then (A5) can be replaced by

$$\sum_{j=1}^{m} \alpha_{m,j} f_j(\,\cdot\,,\,\cdot\,,u) \le K + \epsilon \sum_{j=1}^{m} u_j$$

where $\epsilon > 0$ provided that ϵ is sufficiently small. (However, it is generally crucial that the boundary condition *type* be uniform throughout the system.) For both the Neumann and Robin cases, the operator F in section 2 would be given simply by $Fz = \mathcal{S}(T)z^+$, mapping $C(\overline{\Omega};\mathbb{R}^m)$ into itself, and the set Λ_T would consist of all u_0 satisfying $u_0 = \sigma\mathcal{S}(T)u_0$ with $0 < \sigma < 1$.

Certainly one would like to allow x-dependence in the diffusivities; that is, to have operators of the form $\nabla\cdot(d_i(x,t)\nabla u_i)$ in (1.1). Assuming smoothness and uniform ellipticity, the only obstacle to this is the L^1 estimate in Lemma 3.1. If such an estimate were available, then the remainder of the argument would proceed with only minor modification. This estimate is readily available in the case of Neumann or Robin boundary conditions provided that f satisfies (6.1). One arrives at this estimate by setting $w = \sum_{k=1}^{m} \alpha_{m,k} u_k$ and integrating

$$\frac{\partial w}{\partial t} \;\le\; \nabla\cdot\sum_{k=1}^{m} \alpha_{m,k} d_k \nabla u_k + K - \tilde{\epsilon} w$$

over $\Omega \times (0,T)$ with $u_0 = \sigma u(\,\cdot\,,T)$ and $0 < \sigma < 1$. Here $\tilde{\epsilon} = \epsilon \min\{\alpha_{m,1},\dots,\alpha_{m,m}\}$.

We also remark that if all the diffusivities are equal; *i.e.,* $d_1 = d_2 = \cdots = d_m$, then we need neither the intermediate sums condition (A1) nor the polynomial growth condition (A2) to prove Theorems 1 and 2. Indeed, in this case, global existence follows easily from (A5). By introducing if necessary the additional equation

$$\frac{\partial u_{m+1}}{\partial t} - d_1 \Delta u_{m+1} = K - \sum_{j=1}^{m} \alpha_{m,j} f_j(\,\cdot\,,\,\cdot\,,u)$$

into (1.1) along with zero boundary and initial values, we can assume without loss of generality that $\sum_{j=1}^{m} \alpha_{m,j} f_j(\,\cdot\,,\,\cdot\,,u) = K$, and so $w \equiv \sum_{k=1}^{m} \alpha_{m,k} u_k$ satisfies

$$\begin{aligned} \frac{\partial w}{\partial t} - d_1 \Delta w &= K && \text{in } \Omega \times (0,T) \\ w &= \sum_{k=1}^{m} \alpha_{m,k} g_k && \text{on } \partial\Omega \times (0,T) \\ w(\,\cdot\,,0) &= \sum_{k=1}^{m} \alpha_{m,k} u_{0_k} && \text{on } \overline{\Omega}. \end{aligned}$$

By applying Lemma 4.1 to this scalar equation and by using Lemma 3.1 and the argument in the proof of Lemma 5.1, one finds a $t^* \in (0,T)$ such that $\|w\|_{2,\Omega\times(t^*,T)} \le C$ where C is independent of $u_0 \in \Lambda_T$. From this and Theorem III.8.1 of Ladyženskaja, *et al.* [8], we arrive at the necessary estimate on $\|w(\,\cdot\,,T)\|_{\infty,\Omega}$.

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