PARTLY DISSIPATIVE REACTION-DIFFUSION SYSTEMS AND A MODEL OF PHOSPHORUS DIFFUSION IN SILICON

Selwyn L. Hollis
Department of Mathematics
Armstrong Atlantic State University
Savannah, GA 31419

Jeff J. Morgan
Department of Mathematics
Texas A&M University
College Station, TX 77843

1. Introduction

The following reactions are considered by Richardson and Mulvaney [9] as a model of phosphorus diffusion in silicon:

$$V + I \rightleftharpoons \langle 0 \rangle, \qquad P + V \rightleftharpoons E, \qquad P + I \rightleftharpoons F$$

where P represents a substitutional phosphorus atom, E a phosphorus-vacancy pair, and F a phosphorus-interstitial pair. For more information on the physics, we refer the interested reader to [9] and the references therein. These reactions lead to the reaction-diffusion system

$$\begin{aligned}
V_t - d_1 \Delta V &= -k_1 2PV + k_2 E - k_0 (VI - V_{eq} I_{eq}) \\
E_t - d_2 \Delta E &= k_1 PV - k_2 E \\
I_t - d_3 \Delta I &= -k_3 PI + k_4 F - k_0 (VI - V_{eq} I_{eq}) \\
F_t - d_4 \Delta F &= k_3 PI - k_4 F \\
P_t &= -k_1 PV + k_2 E - k_3 PI + k_4 F
\end{aligned} \tag{1.1}$$

where the k_i are positive reaction rates and the d_i are positive diffusion coefficients. The substitutional phosphorus atom is considered immobile; therefore the equation for P in (1.1) contains no diffusion and is thus an ordinary differential equation satisfied at each x in the domain Ω where the reactions take place. Richardson and Mulvaney [9] take Ω to be the interval $(0, \infty)$ and impose boundary conditions at $x = 0$: $P = C^*, I = I_{eq}, V = V_{eq}$, and $E_x = F_x = 0$. Strictly speaking, it makes no sense to impose a boundary condition on P since there is no diffusion present. However, the boundary condition $P = C^*$ may be *effectively* imposed by requiring: 1) that the initial data $P(x, 0)$ be continuous on $[0, \infty)$ and satisfy $P(0, 0) = C^*$, 2) that $I = I_{eq}, V = V_{eq}, E = k_1/k_2 C^* V_{eq}$, and $F = k_3/k_4 C^* I_{eq}$ hold at $x = 0$, and 3) that the P-equation hold at $x = 0$.

Consider the system (1.1) with diffusion added to the P-equation:

$$\begin{aligned}
V_t^{(\varepsilon)} - d_1 \Delta V^{(\varepsilon)} &= \cdots \\
&\vdots \\
P_t^{(\varepsilon)} - \varepsilon \Delta P^{(\varepsilon)} &= -k_1 P^{(\varepsilon)} V^{(\varepsilon)} + k_2 E^{(\varepsilon)} - k_3 P^{(\varepsilon)} I^{(\varepsilon)} + k_4 F^{(\varepsilon)}
\end{aligned} \tag{1.1$_\varepsilon$}$$

where $\varepsilon > 0$ and we impose Dirichlet boundary conditions on all components. The resulting system $(1.1)_\varepsilon$ fits precisely into the framework of Morgan [7,8] and Hollis and Morgan [4], and thus it is known that solutions of $(1.1)_\varepsilon$ with $\varepsilon > 0$ and bounded, nonnegative initial data exist for all $t \geq 0$. However, two things are not clear:

1) Is the same true for the case $\varepsilon = 0$?

2) Do the solutions of $(1.1)_\varepsilon$ converge in some sense to solutions of (1.1) as $\varepsilon \to 0$?

Indeed, close inspection of the methods used in [4,7,8] reveals that crucial estimates involve constants that depend heavily on the diffusion coefficients and may well grow unbounded as one or more diffusion coefficients tend to zero.

In later work [10], Richardson and Mulvaney consider a more complex model for phosphorus diffusion in silicon which involves charged species:

$$\begin{aligned}
&V^0 + I^0 \rightleftharpoons \langle 0 \rangle, \\
&P^+ + V^- \rightleftharpoons E, \\
&P^+ + I^- \rightleftharpoons F, \\
&V^0 + e^- \rightleftharpoons V^-, \\
&I^0 + e^- \rightleftharpoons I^-.
\end{aligned}$$

This leads to the reaction-diffusion system:

$$\begin{aligned}
V_t^0 - d_1 \Delta V^0 &= -k_1 e^- V^0 + k_2 V^- - k_0 (V^0 I^0 - V_{eq}^0 I_{eq}^0) \\
V_t^- - d_2 \Delta V^- &= k_1 e^- V^0 - k_2 V^- - k_3 P^+ V^- + k_4 E \\
E_t - d_3 \Delta E &= k_3 P^+ V^- - k_4 E \\
I_t^0 - d_4 \Delta I^0 &= -k_5 e^- I^0 + k_6 I^- - k_0 (V^0 I^0 - V_{eq}^0 I_{eq}^0) \\
I_t^- - d_5 \Delta I^- &= k_5 e^- I^0 - k_6 I^- - k_7 P^+ I^- + k_8 F \\
F_t - d_6 \Delta F &= k_7 P^+ I^- - k_8 F \\
P_t^+ &= -k_3 P^+ V^- + k_4 E - k_7 P^+ I^- + k_8 F \\
e_t^- &= -k_1 e^- V^0 + k_2 V^- - k_5 e^- I^0 + k_6 I^- \\
&\quad + k_{neu} (P^+ - V^- - I^-)
\end{aligned} \tag{1.2}$$

where the electrostatic potential has been assumed constant, eliminating terms which model drift of charged species. The last term in the equation for e^- is an empirical term that dynamically enforces local charge neutrality. (We assume there that $k_{neu} \leq \max\{k_2, k_6\}$ so that nonnegativity of solutions is maintained.) In this model, the P^+ and e^- species are considered immobile and thus do not diffuse except through combining with the vacancies or interstitials.

As with system (1.1), if diffusion is introduced into the P^+ and e^- equations, along with boundary conditions of the same type for all eight components, then it is known that solutions exist for all $t \geq 0$, but again the same questions arise concerning the existence of solutions of (1.2) and the convergence of solutions as artificially introduced diffusion coefficients tend to zero.

The structures of these two models motivate us to consider coupled *pairs* of reaction-diffusion *systems* of the form

$$U_t - D\Delta U = f(U,V)$$
$$V_t - \varepsilon\Delta V = g(U,V)$$

where $\varepsilon \geq 0$ and D is a diagonal matrix having fixed, positive diagonal entries. Under suitable hypotheses we will prove global existence of solutions for any $\varepsilon \geq 0$, and convergence and solutions as ε tends to zero. Systems of the above form with $\varepsilon = 0$ are referred to as "partly dissipative" reaction-diffusion systems (cf. Marion [6]).

The investigation of global existence for partly dissipative reaction-diffusion systems and the behavior of solutions of reaction-diffusion systems as certain diffusivities tend to zero may also be motivated through the following examples, adapted from Hollis, Martin, and Pierre [3] and Hollis [2]. Consider the two-component system:

$$\begin{aligned} u_t - u_{xx} &= -uv^\gamma \\ v_t - av_{xx} &= uv^\gamma \end{aligned} \qquad \text{for } x \in (0,1), t > 0. \tag{1.3}$$

where $\gamma > 1$, first with Neumann conditions $u_x = v_x = 0$ imposed at the boundary $x = 0, 1$. If $a > 0$, then it is known (see [2,3,7]) that nonnegative, classical solutions of (1.3) with bounded, nonnegative initial data exist for all $t \geq 0$. However, it is shown in [3] that if $a = 0$ there can be no estimate on $\|v(\cdot,t)\|_\infty$ in terms of $\|u(\cdot,0)\|_\infty$ and $\|v(\cdot,0)\|_\infty$ for large time t.

Now consider (1.3) with (possibly inhomogeneous) Dirichlet conditions imposed at $x = 0, 1$. Again, if $a > 0$, then it is known that nonnegative, classical solutions of (1.3) with bounded, nonnegative initial data exist for all $t \geq 0$. However, it is shown in [1] that if $a = 0$ and a positive Dirichlet condition is imposed on u at $x = 0, 1$, then, for any nonnegative initial data bounded away from zero near the boundary, $\|v(\cdot,t)\|_\infty$ must tend to infinity in finite time. It remains an open question as to whether one can get the same blow-up result with the initial data for v continuous on [0,1] and satisfying $v(0,0) = v(1,0) = 0$. Note that in this case one can consider a zero Dirichlet boundary condition to be effectively imposed on v.

(In the case where $a > 0$ in (1.3) and a positive Dirichlet boundary condition is imposed on u while a homogeneous Neumann boundary condition is imposed on v, it has recently been shown by Bebernes and Lacey [1] that solutions can blow up in finite time if $\gamma > 2$. The question of global existence remains open if $1 < \gamma \leq 2$. However, the difficulty in this case arises from the mixture of boundary condition types, an issue which will not be addressed in this work.)

2. Statement of the General Problem, Hypotheses, and Main Results

Throughout this section, Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. (By "smooth", we mean as usual that $\partial\Omega$ is an $n-1$ dimensional $C^{2+\alpha}$ manifold of which Ω lies locally on one side.)

We consider the reaction-diffusion system:

$$U_t - D\Delta U = F(U) \quad \text{on} \quad \Omega \times (0,\infty) \tag{2.1}$$

where $U = (U_i)_{i=1}^m, F = (F_i)_{i=1}^m$, and $D = \text{diag}(d_1, \ldots, d_m)$ with $d_i \geq 0,\ i = 1, \ldots, m$.

The problem statement is completed by imposing initial and boundary conditions on U. Let us assume for the moment that each $d_i > 0$. We have the initial data:

$$U(x,0) = U_0(x) \geq 0 \quad \text{for} \quad x \in \overline{\Omega} \tag{2.2}$$

and the Dirichlet boundary data:

$$U(x,t) = \Lambda \geq 0 \quad \text{for all} \quad x \in \partial\Omega, t \geq 0. \tag{2.3}$$

U_0 is assumed to be a smooth function on $\overline{\Omega}$ that satisfies the boundary conditions (2.3). When any of the d_i are zero, then of course we cannot impose a boundary condition as such on the corresponding components of U. However, (2.3) remains valid in this situation under the assumption:

(A1) If $d_i = 0$, then $F_i(\Lambda) = 0$ and the ordinary differential equation $\partial_t U_i = F_i(U)$ holds at each $x \in \overline{\Omega}$ and $t > 0$.

This condition *effectively* imposes the boundary condition (2.3) when there are null entries on the diagonal of D. It will not be necessary for us to assume (A1) in order to obtain our global existence results, but we will need (A1) to obtain convergence as artifically introduced diffusion coefficients tend to zero. When (A1) is not assumed, it will be tacitly understood that (2.3) applies only to components of U which correspond to positive d_i and that if $d_i = 0$ the corresponding ordinary differential equation holds at each $x \in \overline{\Omega}$.

The following assumptions are also made on the function F:

(A2) $F \in \mathcal{C}^2(\mathbb{R}^m_+, \mathbb{R}^m)$ and, for each $i = 1, \ldots, m, F_i(\xi) \geq 0$ for all $\xi \in \mathbb{R}^m_+$ with $\xi_i = 0$.

(A3) There exist nonnegative scalars α_{ij} for $i = 1, \ldots, m$ and $j = 1, \ldots, i$ such that $a_{ii} > 0$ for each i and

$$\sum_{j=1}^{i} \alpha_{ij} F_j(\xi) \leq C_1 + C_2 \sum_{j=1}^{m} \xi_j$$

holds for all $\xi \in \mathbb{R}^m_+$, where C_1 and C_2 are constants that are independent of ξ.

(A4) There are constants $C \geq 0$ and $\beta \geq 1$ such that

$$|F_i(\xi)| \leq C\Big[1 + \sum_{j=1}^{m} \xi_j\Big]^{\beta}$$

holds for all $\xi \in \mathbb{R}_+^m$ and $i = 1, \ldots, m$.

Remarks: Assumption (A2) contains smoothness and "quasipositivity" conditions that guarantee local existence of solutions and nonnegativity of solutions as long as they exist. Assumption (A3) is a simple, common form of Morgan's "Intermediate Sums" condition [7,8] which arises naturally in many applications and is used technically in an extension of a duality argument (first applied to two-component reaction-diffusion systems by Hollis, Martin, and Pierre [3]) for obtaining L^p estimates for arbitrary $p \in (1, \infty)$ from a priori L^1 estimates. Finally, assumption (A4) is the usual polynomial growth condition necessary to obtain uniform bounds from p-dependent L^p estimates.

The following local existence result is adapted from Rothe [11; Thm. 1, p. 111]. The reader is referred there for its proof.

Proposition 2.1. *Suppose that the initial data U_0 satisfy the following regularity and compatibility conditions:*

i) $U_{0i} \in \mathcal{C}^{\alpha}(\overline{\Omega})$ for each i with $d_i = 0$;

and for each i with $d_i > 0$,

ii) $U_{0i} \in \mathcal{C}^{2+\alpha}(\overline{\Omega})$,
iii) $U_{0i} = \Lambda_i$ on $\partial\Omega$, and
iv) $-d_i \Delta U_{0i} = F_i(U_0)$ on $\partial\Omega$

where $\alpha \in (0, 1)$. Then under assumption (A2), there exists $T_{\max} \in (0, \infty]$ such that problem (2.1)-(2.3) possesses a unique, nonnegative, noncontinuable, classical solution U on $\overline{\Omega} \times [0, T_{\max})$ satisfying for all $T \in (0, T_{\max})$:

v) $U_i \in \mathcal{C}^{\alpha, 1+\alpha/2}(\overline{\Omega} \times [0, T])$ for each i with $d_i = 0$,
vi) $U_i \in \mathcal{C}^{2+\alpha, 1+\alpha/2}(\overline{\Omega} \times [0, T])$ for each i with $d_i > 0$.

Also, if $U_{0i} \in \mathcal{C}^{2+\alpha}(\overline{\Omega})$ for each i with $d_i = 0$ and if ii)-iv) hold, then

vii) $U_i \in \mathcal{C}^{2+\alpha, 1+\alpha/2}(\overline{\Omega} \times [0, T])$ for each i with $d_i = 0$.

Furthermore,

viii) if $T_{\max} < \infty$, then $\lim_{t \uparrow T_{\max}} \Big\| \sum_{i=1}^{m} U_i(\cdot, t) \Big\|_{\infty, \Omega} = \infty$.

Our next result is for the case where all components of U diffuse. It follows directly from Morgan [7], but in the interest of completeness we provide a proof in Section 3.

Proposition 2.2. *Under assumptions i)-iv) of Proposition 2.1 and assumptions (A2)-(A4), if* $d_i > 0$ *for each* $i = 1, \dots, m$*, then* $T_{\max} = \infty$.

Remark: The regularity and compatibility conditions in Proposition 2.1 are not necessary here. The result is true for $U_0 \in L^\infty(\overline{\Omega}; \mathbb{R}_+^m)$; see [7].

For the situation in which some of the d_i are zero, we make the further assumptions on F:

(A5) The scalars $\alpha_{mj},\ j = 1, \dots, m$, in (A3) may be chosen so that $\alpha_{mj} > 0$ for each j and

$$\sum_{j=1}^{m} \alpha_{mj} F_j(\xi) \le C_1 + C_2 \sum_{d_i > 0} \xi_i$$

for all $\xi \in \mathbb{R}_+^m$.

(A6) There is a scalar $\beta_j > 0$ for each $d_j = 0$ such that

$$\sum_{d_j = 0} \beta_j F_j(\xi) \le C_1 + C_2 \sum_{i=1}^{m} \xi_i$$

for all $\xi \in \mathbb{R}_+^m$.

Proposition 2.3. *Under assumptions i)-iv) of Proposition 2.1 and assumptions (A2)-(A6), if* $d_i \ge 0$ *for each* $i = 1, \dots, m$*, then* $T_{\max} = \infty$.

Let us now partition the system (2.1) into a pair of coupled systems of the form:

$$\begin{cases} u_t - \widetilde{D}\Delta u = f(u,v) \\ v_t = g(u,v) \end{cases} \quad \text{on} \quad \Omega \times (0,\infty)$$

where $u = (u_i)_{i=1}^{\ell}$ and $v = (v_i)_{i=1}^{m-\ell}$, $f = (f_i)_{i=1}^{\ell}$ and $g = (g_i)_{i=1}^{m-\ell}$, and $\widetilde{D} = \operatorname{diag}(\tilde{d}_1, \dots, \tilde{d}_\ell)$ where $\tilde{d}_i > 0$ for $i = 1, \dots, \ell$.

We introduce diffusion into the v-equation above to obtain the system:

$$(2.4)_\varepsilon \qquad \begin{cases} u_t^{(\varepsilon)} - \widetilde{D}\Delta u^{(\varepsilon)} = f(u^{(\varepsilon)}, v^{(\varepsilon)}) \\ v_t^{(\varepsilon)} - \varepsilon \Delta v^{(\varepsilon)} = g(u^{(\varepsilon)}, v^{(\varepsilon)}) \end{cases} \quad \text{on} \quad \Omega \times (0,\infty)$$

together with (relabeled) initial and boundary data from (2.2), (2.3):

$$(2.5)_\varepsilon \qquad \begin{cases} u^{(\varepsilon)}(x,0) = u_0(x) \ge 0 \\ v^{(\varepsilon)}(x,0) = v_0(x) \ge 0 \end{cases} \quad \text{for} \quad x \in \overline{\Omega}$$

$$(2.6)_\varepsilon \qquad \begin{cases} u^{(\varepsilon)}(x,t) = \sigma \ge 0 \\ v^{(\varepsilon)}(x,t) = \rho \ge 0 \end{cases} \quad \text{for} \quad x \in \partial\Omega, t \ge 0$$

Under the assumptions of Proposition 2.2 and 2.3 we know that, for any $\varepsilon \geq 0$, classical solutions of $(2.4)_\varepsilon$-$(2.6)_\varepsilon$ exist for all $t \geq 0$. We now turn our attention to the dependence of solutions on the parameter ε.

First let us note that assumptions (A5) and (A6) may be expressed in terms of u, v, f, and g as

$$(2.7) \qquad \begin{cases} \sum_{j=1}^{\ell} \tilde{\alpha}_{mj} f_j(u,v) + \sum_{j=1}^{m-\ell} \tilde{\alpha}_{m,\ell+j} g_j(u,v) \leq C_1 + C_2 \sum_{i=1}^{\ell} u_i \\ \text{and} \\ \sum_{j=1}^{m-\ell} \tilde{\beta}_j g_j(u,v) \leq C_1 + C_2 \Big[\sum_{j=1}^{\ell} u_j + \sum_{j=1}^{m-\ell} v_j \Big]. \end{cases}$$

We assume further that

(A7) There are nonnegative scalars a_{ij} for $i = 1, \ldots, \ell$ and $j = 1, \ldots, i$ such that $a_{ii} > 0$ for each i and

$$\sum_{j=1}^{i} a_{ij} f_j(\xi, \eta) \leq C_1 + C_2 \sum_{j=1}^{\ell} \xi_j$$

holds for all $\xi \in \mathbb{R}_+^\ell, \eta \in \mathbb{R}_+^{m-\ell}$, and $i = 1, \ldots, \ell$, where C_1 and C_2 are constants independent of ξ and η.

Proposition 2.4. *Under assumptions i)-iv) of Proposition 2.1 as well as (A2)-(A7), $u_i^{(\varepsilon)}$ and $v_j^{(\varepsilon)}$ are bounded in $L^\infty(\Omega \times (0,T))$ independent of ε for each $T \in (0, \infty)$, $i = 1, \ldots, \ell$ and $j = 1, \ldots, m - \ell$.*

Proposition 2.5. *Let the assumptions of Proposition 2.1 be fulfilled, as well as (A1)-(A7). Further assume that $v_0 \in \mathcal{C}^{2+\alpha}(\overline{\Omega}; \mathbb{R}^{m-\ell})$. Then for each $T \in (0, \infty)$, the solution of $(2.4)_\varepsilon$-$(2.6)_\varepsilon$ converges uniformly on $\overline{\Omega} \times [0,T]$ as $\varepsilon \to 0^+$ to the solution of $(2.4)_0$-$(2.6)_0$.*

3. Proofs of the Main Results

Throughout this section Ω will again be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. We assume familiarity on the part of the reader with standard notation, terminology, and theory regarding the Lebesgue and Sobolev spaces $L^p(\Omega), L^p(\Omega \times (0,T))$, and $W_p^{2,1}(\Omega \times (0,T))$ and the usual norms therein. We will use the letter "C" to denote various constants, occasionally emphasizing various dependencies with subscripts or function-suggestive symbols in parentheses. When a constant changes from one estimate to the next, we sometimes use a bar or tilde to indicate this, but we hope that no confusion arises from the use of "$C(T)$", for example, to denote two different constants in consecutive lines.

Consider the system

$$\mu_t - \mathcal{D}\Delta\mu = \mathcal{F}(x,t,\mu) \quad \text{on} \quad \Omega \times (0,\infty) \tag{3.1}$$

where $\mu = (\mu_i)_{i=1}^N, \mathcal{F} = (\mathcal{F}_i)_{i=1}^N$, and $\mathcal{D} = \operatorname{diag}(\delta_1,\ldots,\delta_N)$ with $\delta_i \geq 0$ for each $i = 1,\ldots,N$, together with nonnegative L^∞ initial data $(\mu_{0_i})_{i=1}^N$ and boundary data for each μ_i that corresponds to a positive δ_i:

$$\mu_i(x,t) = z_i \geq 0 \quad \text{for all} \quad x \in \partial\Omega, t > 0, \text{ and } \delta_i > 0. \tag{3.2}$$

Assume further that

(B1) There exist nonnegative scalars α_{ij}, $i = 1,\ldots,N$ and $j = 1,\ldots,i$, such that $\alpha_{ii} > 0$ for each i and

$$\sum_{j=1}^{i} \alpha_{ij}\mathcal{F}_j(x,t,\xi) \leq C_1 + C_2 \sum_{j=1}^{N} \xi_j$$

holds for all $x \in \Omega, t \in \mathbb{R}_+$, and $\xi \in \mathbb{R}_+^N$, where C_1 and C_2 are constants independent of x, t, and ξ. Moreover, α_{Nj}, $j = 1,\ldots,N$ may be chosen so that $\alpha_{Nj} > 0$ if $\delta_j > 0$ and so that

$$\sum_{j=1}^{N} \alpha_{Nj}\mathcal{F}(x,t,\xi) \leq C_1 + C_2 \sum_{\delta_j > 0} \xi_j.$$

Proposition 3.1. *Let $0 < T < \infty$ and suppose that (B1) holds and that a nonnegative, classical solution μ of (3.1)-(3.2) exists for $0 \leq t < T$. If there exists a constant $C(T)$ such that $\left\| \sum_{\delta_i = 0} \mu_i \right\|_{\infty,\Omega\times(0,T)} \leq C(T)$, then for each $p \in [1,\infty)$ there exists a constant $\widehat{C}_p(T)$ such that $\left\| \sum_{i=1}^{N} \mu_i \right\|_{p,\Omega\times(0,T)} \leq \widehat{C}_p(T)$.*

For the sake of completeness, we will give a proof of this result, though the method is essentially the same as that of [4] and [7]. For this we first need some results concerning the following scalar equation.

Let $d > 0, 1 < q < \infty$, and $\theta \in L^q(\Omega \times (0,T))$ with $\theta \geq 0$ and $\|\theta\|_{q,\Omega\times(0,T)} = 1$. Consider the problem

$$\begin{cases} \varphi_t - d\Delta\varphi = \theta & \text{on} \quad \Omega \times (0,T) \\ \varphi = 0 & \text{on} \quad \partial\Omega \times (0,T) \\ \varphi = 0 & \text{on} \quad \Omega \text{ at } t = 0. \end{cases} \tag{3.3}$$

Lemma 3.2. *Problem (3.3) admits a unique solution in $W_q^{2,1}(\Omega \times (0,T))$ satisfying $\varphi \geq 0$ and*

$$\|\varphi\|_{W_q^{2,1}(\Omega\times(0,T))} \leq C_q$$

where C_q is a constant that is independent of θ. Moreover, C_q may be chosen so that:

i) $\|\varphi(\cdot,T)\|_{q,\Omega} \le C_q$;
ii) if $q > \frac{n+2}{2}$, then $\|\varphi\|_{\infty,\Omega\times(0,T)} \le C_q$;
iii) if $1 < q < n+2$ and $q \le r \le \frac{q(n+2)}{n+2-q}$, then $\|\varphi\|_{r,\Omega\times(0,T)} \le C_q$.
iv) if $q > 1$ then $\|\varphi\|_{W_q^{0,1}(\partial\Omega\times(0,T))} \le C_q$

For the proof of Lemma 3.2, we refer to Theorems IV.9.1 and Lemmas II.3.3 and II.3.4 of Ladyženskaja, et al. [5].

For convenience of notation, define $J_+ = \{i\colon\ \delta_i > 0\}$ and $J_0 = \{i\colon\ \delta_i = 0\}$, and let $J_+ = \{j_1, j_2, \ldots, j_s\}$ where $j_{i+1} > j_i$ for $i = 1, \ldots, s-1$. The proof of Proposition 3.1 will be based upon

Lemma 3.3. *Let $0 < T < \infty$, let (B1) be satisfied, and suppose that a nonnegative, classical solution μ of (3.1)-(3.2) exists for $0 \le t < T$. Then there exists a constant $C_p(T)$ such that for each $k \in J_+$:*

(3.4) $$\text{if} \quad 1 < p < \frac{n+2}{n} \quad \text{then}$$

$$\|\mu_k\|_{p,\Omega\times(0,T)} \le C_p(T)\Big[1 + \Big\|\sum_{i=1}^N \mu_i(\cdot,0)\Big\|_{p,\Omega} + \Big\|\sum_{i=1}^N \mu_i\Big\|_{1,\Omega\times(0,T)}\Big]$$

and

(3.5) $$\text{if} \quad \frac{n+2}{n+1} < p < \infty \quad \text{and} \quad r = \frac{p(n+2)}{p+n+2} \quad \text{then}$$

$$\|\mu_k\|_{p,\Omega\times(0,T)} \le C_p(T)\Big[1 + \Big\|\sum_{i=1}^N \mu_i(\cdot,0)\Big\|_{p,\Omega} + \Big\|\sum_{i=1}^N \mu_i\Big\|_{r,\Omega\times(0,T)}\Big].$$

Proof: Take $k \in J_+$ and let φ be the solution of (3.3) with $d = \delta_k$, and set $\bar\theta(\cdot,t) = \theta(\cdot,T-t)$ and $\overline{\varphi}(\cdot,t) = \varphi(\cdot,T-t)$ for $0 \le t \le T$ so that $\overline{\varphi}$ satisfies the backward equation

$$\begin{cases} \overline{\varphi}_t + \delta_k\Delta\overline{\varphi} = -\bar\theta & \text{on} \quad \Omega\times(0,T) \\ \overline{\varphi} = 0 & \text{on} \quad \partial\Omega\times(0,T) \\ \overline{\varphi} = 0 & \text{on} \quad \Omega \text{ at } t = T. \end{cases}$$

Now for each $i = 1, \ldots, k$, integration by parts yields

$$\int_0^T\int_\Omega \mu_i\bar\theta = \int_\Omega \overline{\varphi}(\cdot,0)\mu_i(\cdot,0) + \int_0^T\int_\Omega \overline{\varphi}\mathcal{F}_i + (\delta_k - \delta_i)\int_0^T\int_\Omega \mu_i\Delta\overline{\varphi} - \delta_i\int_0^T\int_\Omega z_i\frac{\partial\overline{\varphi}}{\partial n}.$$

Summing over $i = 1, \dots, k$ and applying (B1) results in

$$\int_0^T \int_\Omega \bar{\theta} \sum_{i=1}^k \alpha_{ki}\mu_i \le \int_\Omega \overline{\varphi}(\cdot,0) \sum_{i=1}^k \alpha_{ki}\mu_i(\cdot,0) + C\int_0^T \int_\Omega \overline{\varphi}\Big[1 + \sum_{i=1}^N \mu_i\Big]$$
$$+ \int_0^T \int_\Omega \Delta\overline{\varphi} \sum_{i=1}^k \alpha_{ki}(\delta_k - \delta_i)\mu_i - \int_0^T \int_{\partial\Omega} \frac{\partial\overline{\varphi}}{\partial n} \sum_{i=1}^k \alpha_{ki}\delta_i z_i,$$

and by Hölder's inequality we obtain

$$\int_0^T \int_\Omega \bar{\theta} \sum_{i=1}^k \alpha_{ki}\mu_i \le C\Big\| \sum_{i=1}^k \mu_i(\cdot,0)\Big\|_{p,\Omega} \|\overline{\varphi}(\cdot,0)\|_{\frac{p}{p-1},\Omega}$$
$$+ C\Big\| 1 + \sum_{i=1}^N \mu_i \Big\|_{r,\Omega\times(0,T)} \|\overline{\varphi}\|_{\frac{r}{r-1},\Omega\times(0,T)}$$
$$+ C\max\{|\delta_k - \delta_i|\}\Big\| \sum_{i=1}^{k-1} \mu_i \Big\|_{p,\Omega\times(0,T)} \|\Delta\overline{\varphi}\|_{\frac{p}{p-1},\Omega\times(0,T)}$$
$$+ C\Big\| \frac{\partial\overline{\varphi}}{\partial n} \Big\|_{\frac{p}{p-1},\partial\Omega\times(0,T)} \tag{3.6}$$

where $p, r \in [1,\infty]$. Now take $q = \frac{p}{p-1}$ and assume first that $q > \frac{n+2}{2}$. Then according to part ii) of Lemma 3.2 we may take $r = 1$ and obtain by duality the estimate

$$\|\mu_k\|_{p,\Omega\times(0,T)} \le C_p(T)\Big[1 + \Big\| \sum_{i=1}^k \mu_i(\cdot,0)\Big\|_{p,\Omega} + \Big\| \sum_{i=1}^N \mu_i \Big\|_{1,\Omega\times(0,T)} + \Big\| \sum_{i=1}^{k-1} \mu_i \Big\|_{p,\Omega\times(0,T)}\Big]$$

for $1 < p < \frac{n+2}{n}$. From this we obtain (3.4) by starting with $k = j_1$ and proceeding by induction on $k = j_2, j_3, \dots, j_s$. To establish (3.5) we take again $q = \frac{p}{p-1}$ but now $r = \frac{p(n+2)}{p+n+2}$, and we assume that $p > \frac{n+2}{n+1}$. A simple calculation reveals that $\frac{r}{r-1} = \frac{p(n+2)}{p(n+1)-(n+2)} = \frac{q(n+2)}{n+2-q}$. Also, $p > \frac{n+2}{n+1}$ is equivalent to $1 < q < n+2$, so we obtain by duality and by part iii) of Lemma 3.2 that

$$\|\mu_k\|_{p,\Omega\times(0,T)} \le C_p(T)\Big[1 + \Big\| \sum_{i=1}^k \mu_i(\cdot,0)\Big\|_{p,\Omega} + \Big\| \sum_{i=1}^N \mu_i \Big\|_{r,\Omega\times(0,T)} + \Big\| \sum_{i=1}^{k-1} \mu_i \Big\|_{p,\Omega\times(0,T)}\Big],$$

from which (3.5) follows by induction on $k = j_1, j_2, \dots, j_s$. ■

The proof of Proposition 3.1, as well as later results, will require

Lemma 3.4. *Let $0 < T < \infty$, let (B1) be satisfied and suppose that a nonnegative, classical solution μ of (3.1)-(3.2) exists for $0 \le t < T$. Then there exists a constant $C(T)$ such that*

$$\int_0^t \sum_{i\in J_+} \mu_i(\cdot,\tau)d\tau \le C(T)$$

for all $t \in [0,T)$.

Proof: Set $w(\cdot,t) = \int_0^t \sum_{j=1}^{N} \alpha_{Nj}\delta_j\mu_j(\cdot,\tau)d\tau$ and observe that w satisfies

$$w_t \leq \delta_{\max}\Big[\Delta w + \sum_{j=1}^{N} \alpha_{Nj}\mu_{0j} + ct + cw\Big]$$

on $\Omega \times (0,T), w = \sum_{\delta_j>0} \alpha_{Nj}\delta_j z_j t$ on $\partial\Omega \times (0,T)$, and $w = \sum_{j=1}^{N} \alpha_{Nj}\delta_j\mu_{0j}$ on Ω at $t=0$. Thus maximum principles imply that $w \leq C(T)$ on $\Omega \times (0,T)$, from which the result follows. ■

We can now proceed with the

Proof of Proposition 3.1. First note that our hypotheses, together with Lemma 3.4, imply that $\Big\| \sum_{i=1}^{N} \mu_i \Big\|_{1,\Omega\times(0,T)} \leq C(T)$. Now (3.4) in Lemma 3.3 provides an estimate $\Big\| \sum_{i=1}^{N} \mu_i \Big\|_{p,\Omega\times(0,T)} \leq \widetilde{C}_p(T)$ for all $p \in \big[1, \frac{n+2}{n}\big)$. Taking $r = \frac{n+2}{n+\tau}$ in (3.5) where $\tau \in (-n,2)$ results in $p = \frac{(n+2)r}{n+2-r} = \frac{n+2}{n+\tau-1}$, and with $\{\tau_j\}_{j=0}^{\infty}$ any decreasing sequence in $(1-n,1]$ such that $\tau_0 = 1$, $\lim_{j\to\infty} \tau_j = 1-n$, and $\tau_j - \tau_{j+1} \leq 1$, we have $r_j < r_{j+1} \leq p_j < p_{j+1} \to \infty$. From (3.4) and (3.5) we then obtain by induction bounds on $\Big\| \sum_{i=1}^{N} \mu_i \Big\|_{p_j,\Omega\times(0,T)}$ for $j = 1,2,3,\ldots$, from which it follows that $\sum_{j=1}^{N} \mu_j \in L^p(\Omega \times (0,T))$ for all $p \in [1,\infty)$. ■

Proposition 3.1 will be the basis for the proofs of Proposition 2.2 and 2.3. Let us now give the

Proof of Proposition 2.2. Since we assume here that each $d_i > 0,\ i = 1,\ldots,m$, in (2.1), assumption (A3) implies that (B1) holds with $\mu = U, \mathcal{D} = D, \mathcal{F} = F$, and thus we have by Proposition 3.1 that $\Big\| \sum_{i=1}^{m} U_i \Big\|_{p,\Omega\times(0,T)} \leq \widehat{C}_p(T)$ for each $p \in [1,\infty)$ and $T \in (0,T_{\max})$ or $T = T_{\max}$ if $T_{\max} < \infty$. Assuming that $T_{\max} < \infty$, we have by (A4) and the Sobolev imbedding theorem that $\Big\| \sum_{i=1}^{m} U_i \Big\|_{\infty,\Omega\times(0,T_{\max})} \leq \widehat{C}(T)$. Thus Proposition 2.1 implies a contradiction from which we conclude that $T_{\max} = \infty$. ■

Proof of Proposition 2.3. This result will again follow from Proposition 3.1, in the same way as did Proposition 2.2, upon the establishment of L^∞ bounds for each U_i with corresponding $d_i = 0$. Here, (A5) implies that (B1) holds with $\mu = U$, etc., and from (A6) we obtain that

$$\sum_{d_j=0} \beta_j U_j(\cdot,t) \leq \sum_{d_j=0} \beta_i U_j(\cdot,0) + c\Big[t + \int_0^t \sum_{j=1}^{m} U_j(\cdot,\tau)d\tau\Big]$$

from which an $L^\infty(\Omega \times (0,T))$ bound on $\sum_{d_j=0} U_j$ follows by Lemma 3.4 and Gronwall's inequality. This estimate holds for $T = T_{\max}$ if $T_{\max} < \infty$, and therefore Proposition 3.1 leads by contradiction to the conclusion that $T_{\max} = \infty$. ■

Let us now turn our attention to problem $(2.4)_\varepsilon$-$(2.6)_\varepsilon$ and the proofs of Proposition 2.4 and 2.5.

Proof of Proposition 2.4. First we apply Proposition 3.1 with $\mu = u^{(\varepsilon)}, \mathcal{D} = \widetilde{D}$, and $\mathcal{F}(x,t,\cdot) = f(\cdot, v^{(\varepsilon)}(x,t))$ and conclude that $\left\| \sum_{i=1}^{\ell} u_i^{(\varepsilon)} \right\|_{p,\Omega\times(0,T)} \le \widehat{C}_p(T)$ for $1 \le p < \infty$ and $0 < T < \infty$ where $\widehat{C}_p(T)$ is independent of ε by assumption (A7). Now set $w = \sum_{j=1}^{m-\ell} \tilde{\beta}_j (v_j^{(\varepsilon)} - \rho_j)$ where $\tilde{\beta}_j$ is from (2.7) and ρ_j from $(2.6)_\varepsilon$, and observe that due to (A6) w satisfies

$$w_t - \varepsilon \Delta w \le C\Big[1 + \sum_{j=1}^{\ell} u_j^{(\varepsilon)} + w\Big] \quad \text{on} \quad \Omega \times (0,T)$$

with $w = 0$ on $\partial\Omega \times (0,T)$. With p an even natural number, multiplying by w^{p-1} and integrating over $\Omega \times (0,t)$ results in

$$\frac{1}{p}\int_\Omega w(\cdot,t)^p \le \frac{1}{p}\int_\Omega w(\cdot,0)^p + c\int_0^t \int_\Omega \Big(1 + \sum_{j=1}^{\ell} u_j^{(\varepsilon)}\Big) w^{p-1} + c\int_0^t \int_\Omega w^p,$$

and application of Young's inequality produces

$$\frac{1}{p}\int_\Omega w(\cdot,t)^p \le \frac{1}{p}\int_\Omega w(\cdot,0)^p + \frac{1}{p}C\int_0^t \int_\Omega \Big(1 + \sum_{j=1}^{\ell} u_j^{(\varepsilon)}\Big)^p + c\frac{2p-1}{p}\int_0^t \int_\Omega w^p. \tag{3.7}$$

Gronwall's inequality now implies that

$$\int_\Omega w(\cdot,t)^p \le \Big[\int_\Omega w(\cdot,0)^p + C \cdot \widehat{C}_p(T)^p\Big] e^{(2p-1)ct} \quad \text{for all} \quad t \in [0,T]$$

from which it follows that

$$\|w(\cdot,t)\|_{p,\Omega} \le \Big[\int_\Omega w(\cdot,0)^p + C \cdot \widehat{C}_p(T)^p\Big]^{1/p} e^{\bar{c}t} \quad \text{for all} \quad t \in [0,T]$$

where $\bar{c}$ is independent of p and the entire right-hand side is independent of ε. It now follows by (A4) and Sobolev imbedding that

$$\Big\| \sum_{i=1}^{\ell} u_i^{(\varepsilon)} \Big\|_{\infty,\Omega\times(0,T)} \le C(T)$$

with $C(T)$ independent of ε. Now returning to (3.7), we obtain by Gronwall's inequality again that

$$\|w(\cdot,t)\|_{p,\Omega} \le \Big[\int_\Omega w(\cdot,0)^p + \widetilde{C}(T)^p\Big]^{1/p} e^{\bar{c}t} \quad \text{for all} \quad t\in[0,T]$$

where $\widetilde{C}(T)$ is independent of p and ε. Letting $p\to\infty$ then results in

$$\|w(\cdot,t)\|_{\infty,\Omega} \le C(T)[\|w(\cdot,0)\|_{\infty,\Omega}+1]e^{\bar{c}t} \quad \text{for all} \quad t\in[0,T].$$

The conclusion of Proposition 2.4 follows. ■

Proof of Proposition 2.5. Let $\varepsilon>0$ and define $\hat{u}^{(\varepsilon)} = u^{(\varepsilon)} - u^{(0)}$ and $\hat{v}^{(\varepsilon)} = v^{(\varepsilon)} - v^{(0)}$ where $(u^{(\varepsilon)}, v^{(\varepsilon)})$ solves $(2.4)_\varepsilon$-$(2.6)_\varepsilon$ and $(u^{(0)}, v^{(0)})$ solves $(2.4)_0$-$(2.6)_0$. Then $(\hat{u}^{(\varepsilon)}, \hat{v}^{(\varepsilon)})$ satisfies

$$(3.8)\qquad \begin{cases} \partial_t\hat{u}_i^{(\varepsilon)} - \tilde{d}_i\Delta\hat{u}_i^{(\varepsilon)} = \Phi_i(\theta_i u^{(\varepsilon)} + (1-\theta_i)u^{(0)}, \theta_i v^{(\varepsilon)} + (1-\theta_i)v^{(0)})\hat{u}^{(\varepsilon)} \\ \qquad + \widetilde{\Phi}_i(\theta_i u^{(\varepsilon)} + (1-\theta_i)u^{(0)}, \theta_i v^{(\varepsilon)} + (1-\theta_i)v^{(0)})\hat{v}^{(\varepsilon)} \\ \partial_t\hat{v}_j^{(\varepsilon)} - \varepsilon\Delta\hat{v}_j^{(\varepsilon)} = \Gamma_j(\tilde{\theta}_j u^{(\varepsilon)} + (1-\tilde{\theta})_j)u^{(0)}, \tilde{\theta}_j v^{(\varepsilon)} + (1-\tilde{\theta}_j)v^{(0)})\hat{u}^{(\varepsilon)} \\ \qquad + \widetilde{\Gamma}_j(\tilde{\theta}_j u^{(\varepsilon)} + (1-\tilde{\theta}_j)u^{(0)}, \tilde{\theta}_j v^{(\varepsilon)} + (1-\tilde{\theta}_j)v^{(0)})\hat{v}^{(\varepsilon)} \\ \qquad + \varepsilon\Delta v_j^{(0)} \text{ for } i=1,\dots,\ell \text{ and } j=1,\dots,m-1 \end{cases}$$

where $\theta_i, \tilde{\theta}_j \in \mathcal{C}(\Omega\times(0,T);[0,1])$; $\Phi_i, \Gamma_j \in \mathcal{C}(\mathbb{R}_+^\ell \times \mathbb{R}_+^{m-\ell}; \mathbb{R}^\ell)$; and $\widetilde{\Phi}_i, \widetilde{\Gamma}_i \in \mathcal{C}(\mathbb{R}_+^\ell \times \mathbb{R}_+^{m-\ell}; \mathbb{R}^{m-\ell})$ are given by the Mean Value Theorem. The initial coniditions $\hat{u}^{(\varepsilon)} = \hat{v}^{(\varepsilon)} = 0$ on $\overline{\Omega}$ at $t=0$ are also satisfied, as well as, because of (A1), the boundary conditions $\hat{u}^{(\varepsilon)} = \hat{v}^{(\varepsilon)} = 0$ on $\partial\Omega\times(0,T)$. By Proposition 2.4, the matrices $\Phi, \widetilde{\Phi}, \Gamma$, and $\widetilde{\Gamma}$, whose rows are from (3.7) the vectors $\Phi_i, \widetilde{\Phi}_i, \Gamma_j$, and $\widetilde{\Gamma}_j$, respectively, have components that are elements of $L^\infty(\Omega\times(0,T))$ with norms bounded independent of ε. (Recall that the functions f and g are $\mathcal{C}^2$.) Taking p to be an even natural number, we can estimate as follows:

$$\begin{aligned} \frac{1}{p}\int_\Omega \sum_{i=1}^{\ell} \hat{u}_i^{(\varepsilon)}(\cdot,t)^p &\le \int_0^t\int_\Omega \sum_{i=1}^{\ell} (\hat{u}_i^{(\varepsilon)})^{p-1}\Big(\sum_{j=1}^{\ell}\Phi_{ij}\hat{u}_j^{(\varepsilon)} + \sum_{j=1}^{m-\ell}\widetilde{\Phi}_{ij}\hat{v}_j^{(\varepsilon)}\Big) \\ &\le C\int_0^t\int_\Omega \sum_{i=1}^{\ell}(\hat{u}_i^{(\varepsilon)})^p + C\int_0^t\int_\Omega\sum_{i=1}^{\ell}|\hat{u}_i^{(\varepsilon)}|^{p-1}\sum_{i=1}^{m-\ell}|\hat{v}_j^{(\varepsilon)}| \\ &\le \frac{2p-1}{p}C\int_0^t\int_\Omega\sum_{i=1}^{\ell}(\hat{u}_i^{(\varepsilon)})^p + \frac{1}{p}C\int_0^t\int_\Omega\sum_{i=1}^{m-\ell}(\hat{v}^{(\varepsilon)})^p. \end{aligned}$$

Gronwall's inequality now yields

$$(3.9)\qquad \sum_{i=1}^{\ell}\|\hat{u}_i^{(\varepsilon)}(\cdot,s)\|_{p,\Omega}^p \le Ce^{(2p-1)ct}\sum_{i=1}^{m-\ell}\|\hat{v}_i^{(\varepsilon)}\|_{p,\Omega\times(0,t)}^p \text{ for } 0\le s\le t\le T$$

in which C is independent of p and ε. Proceeding similarly, we obtain

$$(3.10)\quad \frac{1}{p}\int_\Omega \sum_{i=1}^{m-\ell} \hat{v}_i^{(\varepsilon)}(\cdot,t)^p \le \frac{2p-1}{p} C \int_0^t \int_\Omega \sum_{i=1}^{m-\ell} (\hat{v}_i^{(\varepsilon)})^p + \frac{1}{p} C \int_0^t \int_\Omega \sum_{i=1}^{\ell} (\hat{u}_i^{(\varepsilon)})^p$$
$$+ \frac{1}{p}\varepsilon^p \int_0^t \int_\Omega \sum_{i=1}^{m-\ell} (\Delta v_i^{(0)})^p + \frac{p-1}{p} \int_0^t \int_\Omega \sum_{i=1}^{m-\ell} (\hat{v}_i^{(\varepsilon)})^p,$$

and using (3.9) we have

$$\int_\Omega \sum_{i=1}^{m-\ell} \hat{v}^{(\varepsilon)}(\cdot,t)^p \le [(2p-1)C + p - 1 + C^2 t e^{(2p-1)ct}] \int_0^t \int_\Omega \sum_{i=1}^{m-\ell} (\hat{v}^{(\varepsilon)})^p$$
$$+ \varepsilon^p \int_0^t \int_\Omega \sum_{i=1}^{m-\ell} (\Delta v_i^{(0)})^p.$$

Gronwall's inequality yields

$$(3.11)\qquad \int_\Omega \sum_{i=1}^{m-\ell} \hat{v}_i^{(\varepsilon)}(\cdot,t)^p \le \varepsilon^p \int_0^T \int_\Omega \sum_{i=1}^{m-\ell} (\Delta v_i^{(0)})^p \cdot C_p(T)$$
$$\le \widetilde{C}_p(T) \cdot \varepsilon^p$$

for all $t \in [0,T]$, where $\widetilde{C}_p(T)$ is independent of ε. Together, (3.9) and (3.11) imply an estimate

$$\sum_{i=1}^{\ell} \|\hat{u}_i^{(\varepsilon)}\|_{p,\Omega\times(0,T)} + \sum_{j=1}^{m-\ell} \|\hat{v}_j^{(\varepsilon)}\|_{p,\Omega\times(0,T)} \le C_p(T)\cdot\varepsilon$$

for any $p \in [1,\infty)$, from which Sobolev imbedding results in $\sum_{i=1}^{\ell} \|\hat{u}_i^{(\varepsilon)}\|_{\infty,\Omega\times(0,T)} \le \overline{C}\varepsilon$, the desired result for $\hat{u}^{(\varepsilon)}$. Now returning to (3.10), we see that

$$\int_\Omega \sum_{i=1}^{m-\ell} \hat{v}_i^{(\varepsilon)}(\cdot,t)^p \le [(2p-1)C + p - 1] \int_0^t \int_\Omega \sum_{i=1}^{m-\ell} (\hat{v}_i^{(\varepsilon)})^p$$
$$+ \varepsilon^p \Big[\int_0^t \int_\Omega \sum_{i=1}^{m-\ell} (\Delta v_i^{(0)})^p + \overline{C}^p C|\Omega| t \Big].$$

Thus we conclude that

$$\Big(\int_\Omega \sum_{i=1}^{m-\ell} \hat{v}_i^{(\varepsilon)}(\cdot,t)^p \Big)^{1/p} \le \varepsilon C(T) \quad \text{for all} \quad t \in [0,T]$$

where $C(T)$ is independent of p and ε. Therefore,

$$\sum_{i=1}^{m-\ell} \|\hat{v}_i^{(\varepsilon)}\|_{\infty,\Omega\times(0,T)} \leq \varepsilon C(T),$$

the desired result. ■

4. Examples and Concluding Remarks

Before discussing the phosphorus diffusion problem from Section 1, let us illustrate the applicability of our results by considering the following system which arises from the simple reaction $A + B \rightleftharpoons C$.

$$\begin{aligned} a_t - d_1\Delta a &= -ab + c \\ b_t - d_2\Delta b &= -ab + c \qquad \text{on } \Omega \times (0,\infty) \\ c_t - d_3\Delta c &= ab - c \end{aligned} \tag{4.1}$$

with, for example, Dirichlet boundary data

$$a = b = c = 1 \qquad \text{on } \partial\Omega \times (0,\infty) \tag{4.2}$$

and smooth, nonnegative initial data

$$a = a_0, b = b_0, c = c_0 \qquad \text{on } \Omega \text{ at } t = 0 \tag{4.3}$$

that satisfy the boundary conditions (4.2). Propositions 2.2 and 2.3 imply that solutions exist for all $t \geq 0$ with any combination of zero/positive d_i, with the exception of the case where $d_3 = 0$ and $d_1 + d_2 > 0$ (cf. examples at the end of Section 1). Furthermore, Proposition 2.5 states that solutions converge uniformly on each compact $\overline{\Omega} \times [0,T]$ as: i) $d_i \to 0^+$ with $d_2, d_3 > 0$ fixed, ii) $d_2 \to 0^+$ with $d_1, d_2 > 0$ fixed; iii) $d_1 = d_3 \to 0^+$ with $d_2 > 0$ fixed; iv) $d_2 = d_3 \to 0^+$ with $d_1 > 0$ fixed; v) $d_1 = d_2 = d_3 \to 0^+$.

We now return to the phosphorus diffusion models in Section 1. It is easy to see that these fit precisely into the framework of Section 2. In particular, problem $(1.1)_\varepsilon$, augmented by any set of nonnegative Dirichlet boundary conditions on diffusing components and any smooth, nonnegative initial data, possesses a unique, global, classical, nonnegative solution on $\overline{\Omega} \times [0,\infty)$ for any $\varepsilon \geq 0$ by Propositions 2.2 and 2.3. If the boundary data are

$$V = V_{eq}, I = I_{eq}, E = \frac{k_1}{k_2}C^*V_{eq}, F = \frac{k_3}{k_4}C^*I_{eq}, P = C^* \qquad \text{on } \partial\Omega \times (0,\infty)$$

and the initial data satisfy these boundary conditions, then the solutions are bounded in $L^\infty(\overline{\Omega} \times [0,T])$ independent of ε for each $T \in (0,\infty)$ by Proposition 2.4, and they converge uniformly on $\Omega \times [0,T]$ as $\varepsilon \to 0^+$ to the solution of $(1.1)_0$ by Proposition 2.5.

Analogous results are also valid for the system (1.2) with diffusion terms $\varepsilon\Delta P^+$ and $\varepsilon\Delta e^-$ added to the corresponding equations. We refer to this modified system as $(1.2)_\varepsilon$. For any set of nonnegative Dirichlet boundary data imposed on diffusing species, and any smooth, nonnegative initial data, $(1.2)_\varepsilon$ possesses a unique, global, classical, nonnegative solution on $\overline{\Omega}\times[0,\infty)$ for any $\varepsilon \geq 0$. If the boundary data give rise to a constant equilibrium (i.e., all reaction functions are zero on $\partial\Omega$) and if the initial data are smooth and satsify the boundary conditions, then solutions are bounded in $L^\infty(\overline{\Omega}\times[0,T])$ independent of ε for each $T \in (0,\infty)$ and converge uniformly on $\overline{\Omega}\times[0,T]$ as $\varepsilon \to 0^+$ to the solution of $(1.2)_0$.

We remark here that the global existence results of Propositions 2.2 and 2.3 and the boundedness result of Proposition 2.4 remain valid for problem (2.1)-(2.3) if the Dirichlet boundary conditions (2.3) are replaced by Robin/Neumann Conditions: $\frac{\partial}{\partial n}U_i = \gamma_i[\Lambda_i - U_i]$ on $\partial\Omega\times(0,\infty)$ for each i with $d_i > 0$ and where $\gamma_i, \Delta_i \in [0,\infty)$. The proofs require only straightforward modification. It is important that the boundary condition *type* be uniform throughout the diffusing components, i.e., either Dirichlet throughout or Robin/Neumann throughout. Serious technical difficulties arise if Robin/Neumann conditions are imposed on some components while inhomogeneous Dirichlet conditions are imposed on others (cf. remarks concluding Section 1). A notable exception to this, however, is that in $(2.4)_\varepsilon$-$(2.6)_\varepsilon$ one may impose one type of boundary condition throughout the components of $u^{(\varepsilon)}$ and another throughout $v^{(\varepsilon)}$ (when $\varepsilon > 0$) with little difficulty in modifying the arguments. The rather simple-minded argument in the proof of the convergence result of Proposition 2.5 relies heavily, however, upon having Dirichlet conditions on the $v_j^{(\varepsilon)}$, so that we could guarantee easily that $v_j^{(\varepsilon)} - v_j^{(0)}$ satisfied a *homogeneous* boundary condition on $\partial\Omega$.

The global existence results of Propositions 2.2 and 2.3 also remain valid in the case where Ω is an unbounded domain such as a half-space as considered in [9,10]. The proofs can be modifed using the interior estimate ideas in [4]. It is not clear, however, whether (appropriate analogs of) Propositions 2.4 and 2.5 are true in this case. Also, the methods of [4] provide a convenient method for obtaining the necessary estimates for the proofs of the global existence results in the case where $\Omega = (0,L)$, $L < \infty$, and Dirichlet conditions are imposed at $x = 0$ and Robin/Neumann conditions are imposed at $x = L$.

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