INTERIOR ESTIMATES FOR A CLASS OF REACTION-DIFFUSION SYSTEMS FROM L^1 A PRIORI ESTIMATES

Selwyn L. Hollis
Department of Mathematics
Armstrong Atlantic State University
Savannah, Georgia 31419 USA

Jeff Morgan†
Department of Mathematics
Texas A&M University
College Station, Texas 77843 USA

Abstract

We obtain interior estimates for a class of semilinear reaction-diffusion systems from L^1 a priori estimates. Our results are applied to a predator-prey model in which the species switch the role of predator and prey on given subsets of their domain of interaction, and a one dimensional flame propagation model. Extensions of earlier results in Morgan [14], [15] follow from the analysis.

1. Introduction

In the past ten years, a large body of literature has been generated from a global existence problem posed by Professor R. H. Martin. The problem was to determine whether solutions v_1, v_2 of

$$(1.1)\qquad \begin{cases} v_{1_t} = d_1 \Delta v_1 - v_1 v_2^\gamma, & t > 0,\ x \in \Omega \\ v_{2_t} = d_2 \Delta v_2 + v_1 v_2^\gamma, & t > 0,\ x \in \Omega, \end{cases}$$

subject to various boundary conditions and nonnegative initial data, exist globally (i.e. for all $t > 0$). For $d_1, d_2 > 0$ and homogeneous boundary conditions, Alikakos [1] proved that solutions of (1.1) exist globally provided that $1 \leq \gamma < \frac{n+2}{n}$ (here $\Omega \subseteq \mathbb{R}^n$ is a bounded domain with smooth boundary). Later, Masuda [13] proved global existence and uniform boundedness on $(0, \infty) \times \Omega$ if $d_1, d_2 > 0$ and $\gamma \geq 1$. Since that time, global existence results have been given for an entire class of systems which include (1.1) (see Hollis, Martin and Pierre [9], Morgan [14], Haraux and Youkana [8], Kanel' [10] and the references therein).

---

† Supported in part by NSF Grant # DMS-8813071

However, unless γ is small, there are no results in the literature which can be applied to the following modification of (1.1). Consider

$$\tag{1.2}\begin{cases} v_{1_t} = d_1\Delta v_1 + a_1 v_1 + b_1 v_2 - c(x) v_1 v_2^\gamma & t>0,\ x\in\Omega \\ v_{2_t} = d_2\Delta v_2 + a_2 v_1 + b_2 v_2 + c(x) v_1 v_2^\gamma & t>0,\ x\in\Omega \end{cases}$$

subject to

$$\tag{1.3}\begin{cases} v_1 = v_2 = 0 & t>0,\ x\in\partial\Omega \\ v_1, v_2 \ge 0 & t=0,\ x\in\Omega \end{cases}$$

where $d_1, d_2 > 0;\ b_1, a_2 \ge 0;\ a_1, b_2 \in \mathbb{R}$ and $c \in C(\overline{\Omega}, [-1,1])$. Such a model could arise if two species were interacting on a given domain and switched the roles of predator and prey depending on their locations within the domain. We note that a priori bounds can be obtained for v_1, v_2. Clearly, if we sum the equation in (1.2) and integrate over Ω, then we can obtain bounds in $L^1(\Omega)$.

It is also possible to modify results in [14] to obtain $L^2((0,T)\times\Omega)$ estimates for v_1 and v_2. Still, unless $d_1 = d_2,\ 1+\gamma < \frac{n+2}{n}$, or c is of one algebraic sign, global existence does not follow.

In this note we are motivated by (1.2) and (1.3) to obtain interior estimates for m component functions which satisfy a fairly general system of parabolic inequalities. Since the proofs in the general setting are quite technical and admittedly unattractive, it might be helpful to look at a simple setting and outline some proofs.

For the sake of completeness, we first consider (1.1) subject to the conditions (1.3). Then we certainly have $v_1, v_2 \ge 0$, and if $\varphi(t,x)$ is smooth with $\varphi|_{\partial\Omega} \equiv 0$, then integration by parts yields

$$\tag{1.4}\begin{aligned} -\int_0^T\int_\Omega v_1[\varphi_t + d_1\Delta\varphi]\,dxdt + \int_\Omega v_1(T,x)\varphi(T,x)dx - \int_\Omega v_1(0,x)\varphi(0,x)dx \\ = \int_0^T\int_\Omega v_2[\varphi_t + d_2\Delta\varphi]\,dxdt - \int_\Omega v_2(T,x)\varphi(T,x)dx \\ + \int_\Omega v_2(0,x)\varphi(0,x)dx. \end{aligned}$$

Now, given $\theta \in L^q((0,T)\times\Omega)$ with $1<q<\infty$ choose φ so that

$$\begin{aligned} &\varphi_t + d_2\Delta\varphi = -\theta \quad \text{on} \quad (0,T)\times\Omega \\ &\varphi = 0 \quad \text{on} \quad (0,T)\times\partial\Omega \\ &\varphi(T,\cdot) = 0 \quad \text{on} \quad \Omega. \end{aligned}$$

Then (1.4) takes the form

$$
\begin{aligned}
(1.5)\qquad & -\int_0^T\int_\Omega v_1[\varphi_t+d_1\Delta\varphi]dxdt-\int_\Omega v_1(0,x)\varphi(0,x)dx\\
&\qquad =-\int_0^T\int_\Omega v_2\theta dxdt+\int_\Omega v_2(0,x)\varphi(0,x)dx.
\end{aligned}
$$

Furthermore, from maximum principles, v_1 is uniformly bounded, and by parabolic regularity theory we have strong estimates of φ in terms of θ. Thus, (1.5) yields

$$
(1.6)\qquad \int_0^T\int_\Omega v_2\theta dxdt\le C\|\theta\|_{L^q((0,T)\times\Omega)}.
$$

Consequently we obtain $\|v_2\|_{L^p((0,T)\times\Omega)}\le C$ where $p=\frac{q}{q-1}$. By choosing p arbitrarily large and taking advantage of the polynomial nature of the reaction terms in (1.1), we can again employ parabolic regularity to obtain a sup-norm bound for v_2.

Now we consider (1.2) subject to (1.3), and for simplicity assume $a_1=b_1=a_2=b_2=0$. Suppose $c(x)$ is two-sided. That is, if we define $\Omega_+=\{x\in\Omega\mid c(x)>0\}$ and $\Omega_-=\{x\in\Omega\mid c(x)<0\}$, then Ω_+ and Ω_- are both nonempty. We will demonstrate that if $S=\{x\in\Omega\mid c(x)=0\}$ and finite time blow-up occurs for the solution of (1.2), (1.3), then it must occur "near" S. For example, if $\Omega=(0,1)$ and $c(x)>0$ for $x<1/2$, $c(x)<0$ for $x>1/2$ and $0<\varepsilon<1/2$, then we can obtain sup-norm bounds for v_1 and v_2 (dependent upon ε) on sets of the form $\{x\mid |x-\frac{1}{2}|>\varepsilon,\ 0\le x\le 1\}$. We outline our proof as follows.

Again we have $v_1,v_2\ge 0$. Also, as mentioned earlier, we can obtain an L^1 estimate for v_1 and v_2. Then since

$$
(1.7)\qquad v_{1_t}\le d_1\Delta v_1\quad\text{on}\quad (0,T)\times\Omega_+,
$$

the nonnegativity and the L^1 estimate on v_1 imply a sup-norm bound for v_1 on any subdomain of Ω_+. Let $\varepsilon>0$ and suppose $\Omega_{2\varepsilon}\subset\Omega_\varepsilon\subset\Omega_+$ are smooth domains such that $\mathrm{dist}(\Omega_\varepsilon,\Omega_-)\ge\varepsilon$ and $\mathrm{dist}(\Omega_{2\varepsilon},\Omega_\varepsilon^c)\ge\varepsilon$. Furthermore, let $\varphi(t,x)\ge 0$ be smooth with $\varphi|_{\partial\Omega_\varepsilon}\equiv 0$. Then integration by parts yields

$$
\begin{aligned}
(1.8)&\\
&-\int_0^T\int_{\Omega_\varepsilon} v_1[\varphi_t+d_1\Delta\varphi]dxdt+\int_{\Omega_\varepsilon} v_1(T,x)\varphi(T,x)dx-\int_{\Omega_\varepsilon} v_1(0,x)\varphi(0,x)dx\\
&\qquad\le\int_0^T\int_{\Omega_\varepsilon} v_2[\varphi_t+d_2\Delta\varphi]dxdt-\int_{\Omega_\varepsilon} v_2(T,x)\varphi(T,x)dx\\
&\qquad\quad+\int_{\Omega_\varepsilon} v_2(0,x)\varphi(0,x)dx-\int_0^T\int_{\partial\Omega_\varepsilon}(d_1v_1+d_2v_2)\frac{\partial\varphi}{\partial n}.
\end{aligned}
$$

This last boundary integral may be made to disappear, thereby introducing an additional manageable term, if we multiply first by a suitable smooth function which vanishes on $\partial\Omega_\varepsilon$ and is 1 on $\Omega_{2\varepsilon}$. So, if we let $\theta \in L^q((0,T) \times \Omega_\varepsilon)$, $\theta \geq 0$ for $1 < q < \infty$ and take φ to be the solution of

$$\begin{aligned}
&\varphi_t + d_2 \Delta\varphi = -\theta \quad \text{on} \quad (0,T) \times \Omega_\varepsilon \\
&\varphi = 0 \quad \text{on} \quad (0,T) \times \partial\Omega_\varepsilon \\
&\varphi(T,\cdot) = 0 \quad \text{on} \quad \Omega_\varepsilon,
\end{aligned}$$

then similarly to (1.6) we obtain $v_2 \in L^p((0,T) \times \Omega_{2\varepsilon})$ for all $1 < p < \infty$, and interior estimates for parabolic equations can be used to obtain sup-norm bounds for v_2 on subdomains of $\Omega_{2\varepsilon}$. Similar results can be found on subdomains of Ω_- by interchanging the rolls of v_1 and v_2 above. The details of these arguments can be found in section 3.

In section 4 we apply our results to a model for one dimensional flame propagation. We also discuss how these results can be used to extend some earlier global existence, boundedness and decay results in [14], [15] to unbounded domains.

We are indebted to the referee for suggestions which improved the readability of this paper.

2. Notation and Statement of Some L^p Results

We assume throughout that $0 < T \leq \infty$; $m, n \in \mathbf{N}$ and $\Omega \subseteq \mathbb{R}^n$ is a domain. If $\Omega \neq \mathbb{R}^n$ then we assume that the boundary of Ω, denoted $\partial\Omega$, is a $C^{2+\alpha}$ manifold such that Ω lies locally on one side of $\partial\Omega$. Define $\mathbb{R}^n_+ = \{x \mid x \in \mathbb{R}^n, x_i \geq 0 \text{ for all } 1 \leq i \leq n\}$. For each $i, j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, m\}$ let $a^k_{ij} \in C^{0,2}(\mathbb{R}_+ \times \overline{\Omega}, \mathbb{R})$, $c^k_i \in C^{0,1}(\mathbb{R}_+ \times \overline{\Omega}, \mathbb{R})$, and $f_k \in C([0,T) \times \overline{\Omega}, \mathbb{R})$, and let

$$\mathcal{L}_k = \sum_{i,j=1}^{n} a^k_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{\ell=1}^{n} c^k_\ell \frac{\partial}{\partial x_\ell}$$

be a uniformly elliptic operator. That is, there exists $\alpha > 0$ such that $\sum\limits_{i,j=1}^{n} a^k_{i,j} \xi_i \xi_j \geq \alpha \|\xi\|^2$ for all $\xi \in \mathbb{R}^n$.

We consider the semilinear parabolic system of partial differential inequalities

$$\text{(2.1)} \qquad \begin{cases} u_{k_t} \leq \mathcal{L}_k u_k + f_k, \qquad 0 < t < T, \ x \in \Omega \\ u_k \geq 0, \quad 0 \leq t < T, \ x \in \overline{\Omega}, \end{cases} \qquad k \in \{1, \ldots, m\}.$$

We say that $u = (u_k)$ is a solution of (2.1) if $u = (u_k) \in C^{1,2}((0,T) \times \Omega, \mathbb{R}^m_+) \cap C([0,T) \times \Omega, \mathbb{R}^m_+)$ and satisfies (2.1).

Now let $\Omega' \subseteq \Omega$ be a bounded domain and suppose $u = (u_k)$ is a solution of (2.1). We say that (2.1) satisfies a Total Summing Condition with respect to u and Ω' if the following three conditions are satisfied.

(S1) There exists a bounded domain $\Omega'' \subseteq \Omega$ with $C^{2+\alpha}$ boundary such that Ω'' lies locally on one side of $\partial\Omega''$ with $\Omega' \subseteq \Omega''$ and $\mathrm{dist}(\Omega', \Omega \backslash \Omega'') > 0$.

(S2) If $\partial\Omega' \cap \partial\Omega = \emptyset$ then $\partial\Omega'' \cap \partial\Omega = \emptyset$, and if $\partial\Omega' \cap \partial\Omega \neq \emptyset$ then for all $k \in \{1, \ldots, m\}$ there exist $b_k \in C(\mathbb{R}_+ \times (\partial\Omega'' \cap \partial\Omega), \mathbb{R}_+)$ such that $u_k \leq b_k$ on $(0, T) \times (\partial\Omega'' \cap \partial\Omega)$.

(S3) For all $i, j \in \{1, \ldots, m\}$ such that $i \geq j$ there exist $\alpha_{ij} \in \mathbb{R}_+$ with $\alpha_{ii} > 0$ and $M_1, M_2 \in C(\mathbb{R}_+, \mathbb{R})$ such that for all $k \in \{1, \ldots, m\}$

$$\sum_{j=1}^{k} \alpha_{kj} f_j(t, x) \leq M_1(t) \sum_{i=1}^{m} u_i(t, x) + M_2(t)$$

for all $(t, x) \in (0, T) \times \Omega''$.

Perhaps some remarks are in order at this point. Consider (1.2)–(1.3) with $\Omega = (0, 1)$; $v_1(0, \cdot), v_2(0, \cdot) \in C_0([0, 1], \mathbb{R}_+)$ and $c \in C^{\alpha}([0, 1], \mathbb{R})$. Then well known results imply that there exists $T_{\max} > 0$ and a classical noncontinuable solution v_1, v_2 of (1.2)–(1.3) on $[0, T_{\max}) \times \overline{\Omega}$. In addition, v_1, v_2 are nonnegative. Now, suppose $c(x) > 0$ for $0 < x < \frac{1}{2}$ and let $0 < \varepsilon < \frac{1}{2}$. Let $m = 2,\ T = T_{\max},\ u_1 = v_1,\ u_2 = v_2,\ f_1 = a_1 v_1 + b_1 v_2 - c v_1 v_2^{\gamma}$ and $f_2 = a_2 v_1 + b_2 v_2 + c v_1 v_2^{\gamma}$. If $\Omega' = (0, \frac{1}{2} - \varepsilon)$ then we can easily verify that (2.1) satisfies a Total Summing Condition with respect to u and Ω'. Similarly, if $c(x) < 0$ for $\frac{1}{2} < x < 1$ and we set $u_1 = v_2,\ u_2 = v_1,\ f_1 = a_2 v_1 + b_2 v_2 + c v_1 v_2^{\gamma}$ and $f_2 = a_1 v_1 + b_1 v_2 - c v_1 v_2^{\gamma}$, then again we can easily verify that (2.1) satisfies a Total Summing Condition with respect to u and $\Omega' = (\frac{1}{2} + \varepsilon, 1)$. We will give further remarks in section 4, including the relation of (2.1) and the Total Summing Condition to the separable generalized Lyapunov structure and intermediate sum condition in [14], [15].

We are now in a position to state some results.

Proposition 1: Suppose that $0 < T < \infty$, u is a solution of (2.1) and $\Omega' \subseteq \Omega$ is a bounded domain such that (2.1) satisfies a Total Summing Condition with respect to u and Ω'. If Ω'' is given in (S1), $\|u\|_{1,(0,T)\times\Omega''} < \infty$ and $1 < p < \infty$, then $\|u\|_{p,(0,T)\times\Omega'} < \infty$.

We can also give an L^p boundedness and decay result in the case $T = \infty$.

Proposition 2: Suppose $T = \infty$, u is a solution of (2.1) and $\Omega' \subseteq \Omega$ is a bounded domain such that (2.1) satisfies a Total Summing Condition with respect to u and Ω'. Let

$1 < p < \infty$ and Ω'' be given in (S1). If there exists $K_1 > 0$ such that for all $k \in \{1, \ldots, m\}$, $i, j \in \{1, \ldots, n\}$ and $\ell \in \{1, 2\}$ we have $\|b_k\|_\infty, \|M_\ell\|_\infty, \|a_{ij}^k\|_{C^{1,2}}, \|c_i^k\|_{C^{0,1}}, \|u\|_{1,(t,t+1)\times\Omega''} \leq K_1$ for all $t \geq 0$, then $\limsup_{t\to\infty} \|u\|_{p,(t,t+1)\times\Omega'} < \infty$. If in addition $\|b_k(t,\cdot)\|_{\infty,\partial\Omega''\cap\partial\Omega}$, $M_2(t), \|u\|_{1,(t,t+1)\times\Omega''} \to 0$ as $t \to \infty$ then $\|u\|_{p,(t,t+1)\times\Omega'} \to 0$ as $t \to \infty$.

We give the proofs of Propositions 1 and 2 in section 3. Section 4 contains some extensions of these results to systems with Neumann type inequalities in condition (S2) and a result which extends these results to sup norm bounds and decay. We also discuss how our analysis can be used to extend certain results in [14], [15] to unbounded domains.

3. Proofs of Propositions 1 and 2

The primary technique employed in this section is an extension of a duality argument which was introduced in Hollis, Martin and Pierre [9] and used extensively in Morgan [14], [15].

Throughout this section we assume that the hypotheses of Proposition 1 are satisfied. Define $\Omega_0 = \Omega''$ from (S1), and for $k \in \mathbb{N}$ let $\Omega_k \subseteq \Omega$ be a bounded domain with $C^{2+\alpha}$ boundary such that Ω_k lies locally on one side of $\partial\Omega_k$. Furthermore, suppose that $\Omega' \subseteq \Omega_k \subseteq \Omega_{k-1} \subseteq \Omega$ with $\text{dist}(\Omega', \Omega\backslash\Omega_k) > 0$ and $\text{dist}(\Omega_k, \Omega\backslash\Omega_{k-1}) > 0$. For each $k \in \mathbb{N}$ let $g_k \in C_0^\infty(\mathbb{R}^n, [0,1])$ such that $g_k|_{\Omega_k} \equiv 1$ and $g_k|_{\Omega\backslash\Omega_{k-1}} \equiv 0$.

Let $1 < p < \infty,\ \tau \geq 0,\ s > 0,\ k \in \{1, \ldots, m\}$ and suppose that $\theta \in L^p((\tau, \tau+s) \times \Omega_0, \mathbb{R}_+)$ is such that $\|\theta\|_{p,(\tau,\tau+s)\times\Omega_0} = 1$. Define

$$\widetilde{\mathcal{L}}_k = \sum_{i,j=1}^n a_{i,j}^k \frac{\partial^2}{\partial x_i \partial x_j} - \sum_{\ell=1}^n c_\ell^k \frac{\partial}{\partial x_\ell}.$$

We will make considerable use of the following scalar equation.

$$\text{(3.1)} \qquad \begin{cases} \phi_t = \widetilde{\mathcal{L}}_k \phi + \theta, & \tau < t < \tau + s,\ x \in \Omega_0 \\ \phi = 0, & \tau < t < \tau + s,\ x \in \partial\Omega_0 \\ \phi = 0, & t = \tau,\ x \in \Omega_0. \end{cases}$$

Our first Lemma gives well known estimates for the solution of (3.1) which are critical in the proof of Propositions 1 and 2. We refer the reader to Ladyženskaja et al. [11] for this result.

Lemma 3.1: There exists a unique solution $\phi \in W_p^{1,2}((\tau, \tau+s) \times \Omega_0)$ of (3.1). Furthermore, $\phi \geq 0$ and there exists $C > 0$ depending on $\Omega_0, s, p, \alpha, \|a_{ij}^k\|_{C^1([\tau,\tau+s]\times\overline{\Omega}_0)}$ and $\|c_\ell^k\|_{\infty,(\tau,\tau+s)\times\Omega_0}$ and independent of θ such that

$$\|\phi\|_{W_p^{1,2}((\tau,\tau+s)\times\Omega_0)} \leq C.$$

In addition, C can be chosen such that the following are true:

(i) if $p > n+2$ then $\||\nabla\phi|\|_{\infty,(\tau,\tau+s)\times\Omega_0} \le C$;
(ii) if $p > \frac{n+2}{2}$ then $\|\phi\|_{\infty,(\tau,\tau+s)\times\Omega_0} \le C$;
(iii) if $1 < p < n+1$ and $p \le q \le \frac{np}{n+1-p}$ then $\|\phi(\tau+s,\cdot)\|_{q,\Omega_0} \le C$;
(iv) if $1 < p < n+2$ and $p \le q \le \frac{p(n+2)}{n+2-p}$ then $\|\phi\|_{W_q^{0,1}((\tau,\tau+s)\times\Omega_0)} \le C$;
(v) if $p > 1$ then $\|\phi\|_{W_p^{0,1}((\tau,\tau+s)\times\partial\Omega_0)} \le C$.

The following technical Lemma forms the core of the proofs of Propositions 1 and 2.

Lemma 3.2: Suppose $1 \le a < \infty$, $\tau + s \le T$ (with strict inequality if $T = \infty$) and $q \in \mathbb{N}$ such that $u \in L^a((\tau,\tau+s)\times\Omega_{q-1},\mathbb{R}_+^m)$ and $u(\tau,\cdot) \in L^a(\Omega_{q-1},\mathbb{R}_+^m)$. If $p \in (1,\infty)$ is chosen such that $\|\phi(\tau+s,\cdot)\|_{\frac{a}{a-1},\Omega_0}, \|\phi\|_{\frac{a}{a-1},(\tau,\tau+s)\times\Omega_0}, \||\nabla\phi|\|_{\frac{a}{a-1},(\tau,\tau+s)\times\Omega} \le C$, where C is given in Lemma 3.1, then there exists $L > 0$, dependent upon $\|g_q\|_{C^2(\overline{\Omega}_0)}$, p, Ω_0, s, $\|a_{ij}^k\|_{C^{1,2}([\tau,\tau+s]\times\overline{\Omega}_0)}$, $\|c_\ell^k\|_{C^{0,1}([\tau,\tau+s]\times\overline{\Omega}_0)}$ and $\|M_1\|_{\infty,(\tau,\tau+s)}$, such that $u_h \in L^{p/(p-1)}((\tau,\tau+s)\times\Omega_q)$ and

$$\begin{aligned}\|u_h\|_{\frac{p}{p-1},(\tau,\tau+s)\times\Omega_q} \le L[&\|u(\tau,\cdot)\|_{a,\Omega_{q-1}} + \|u\|_{a,(\tau,\tau+s)\times\Omega_{q-1}} \\ &+ \|b\|_{\infty,(\tau,\tau+s)\times(\partial\Omega''\cap\partial\Omega)} + \|M_2\|_{\infty,(\tau,\tau+s)}].\end{aligned}$$

for all $h \in \{1,\ldots,m\}$.

Proof: Let ϕ be the unique solution of (3.1) and set $\tilde{\phi}(t,x) = \phi(2\tau+s-t,x)$ and $\tilde{\theta}(t,x) = \theta(2\tau+s-t,x)$ for all $\tau \le t \le \tau+s$ and $x \in \overline{\Omega}_0$. Then $\tilde{\phi}$ satisfies

$$\begin{array}{ll}\tilde{\phi}_t = -\widetilde{\mathcal{L}}_k\tilde{\phi} - \tilde{\theta}, & \tau < t < \tau+s,\ x \in \Omega_0 \\ \tilde{\phi} = 0, & \tau < t < \tau+s,\ x \in \partial\Omega_0 \\ \tilde{\phi} = 0, & t = \tau+s,\ x \in \Omega_0.\end{array}$$

Consequently, for all $1 \le h \le k$, integration by parts yields

$$\begin{aligned}(3.2)\qquad \int_\tau^{\tau+s}\int_{\Omega_{q-1}} u_h g_q \tilde{\theta}\,dxdt \le{}& \int_{\Omega_{q-1}} u_h(\tau,x) g_q(x)\tilde{\phi}(\tau,x)dx \\ &- \int_\tau^{\tau+s}\int_{\partial\Omega''\cap\partial\Omega} u_h \sum_{i,j=1}^n \frac{\partial}{\partial x_j}(\tilde{\phi}a_{i,j}^h)\eta_i d\sigma dt \\ &+ \int_\tau^{\tau+s}\int_{\Omega_{q-1}} u_h \sum_{i,j=1}^n \Big(\tilde{\phi}\frac{\partial^2(g_q a_{ij}^h)}{\partial x_i\partial x_j} + \frac{\partial\tilde{\phi}}{\partial x_i}\frac{\partial(g_q a_{ij}^h)}{\partial x_j}\Big)dxdt \\ &- \int_\tau^{\tau+s}\int_{\Omega_{q-1}} u_h \sum_{\ell=1}^n \tilde{\phi}\frac{\partial(g_q c_\ell^h)}{\partial x_\ell}dxdt + \int_\tau^{\tau+s}\int_{\Omega_{q-1}} \tilde{\phi} g_q f_h dxdt \\ &+ \int_\tau^{\tau+s}\int_{\Omega_{q-1}} [(\widetilde{\mathcal{L}}_h - \widetilde{\mathcal{L}}_k)\tilde{\phi}]u_h g_q dxdt.\end{aligned}$$

Now, (S3) yields

$$
\begin{aligned}
(3.3)\quad \int_{\tau}^{\tau+s}\int_{\Omega_{q-1}}\sum_{h=1}^{k}\alpha_{kh}u_h g_q\tilde{\theta}\,dxdt \le{}& \sum_{h=1}^{k}\alpha_{kh}[\text{first 4 terms RHS (3.2)}]\\
&+\int_{\tau}^{\tau+s}\int_{\Omega_{q-1}}\tilde{\phi}g_q\left[M_1(t)\sum_{i=1}^{m}u_i+M_2(t)\right]dxdt\\
&+\sum_{h=1}^{k-1}\alpha_{kh}\int_{\tau}^{\tau+s}\int_{\Omega_{q-1}}[(\widetilde{\mathcal{L}}_h-\widetilde{\mathcal{L}}_k)\tilde{\phi}]u_h g_q\,dxdt
\end{aligned}
$$

We now apply Hölder's inequality, (S2), Lemma 3.1, and the hypotheses of Lemma 3.2 and conclude by duality that the result of Lemma 3.2 holds when $k = 1$. Now suppose there exists $k \in \{1,\ldots,m\}$ such that the result is true for $1 \le h < k \le m$. Then if we isolate the k^{th} term in the sum on the left hand side of (3.3), repeat the above argument and use our supposition that the result of Lemma 3.2 holds for $1 \le h < k$, then duality gives the result for $h = k$ as well. Hence, the result holds for all $h \in \{1,\ldots,m\}$. ■

We are now in a position to obtain L^p estimates for u. Unless stated otherwise, all bounding constants given below depend upon $\Omega_0, T, a_{ij}^k, c_\ell^k, M_1$ and M_2 as well as their stated dependencies. We first consider the case $T < \infty$.

Lemma 3.3: Suppose $0 < T < \infty$. Let $1 \le \delta < \frac{n+2}{n+1}$. Then there exists $L_1 > 0$ dependent upon δ and $\|g_1\|_{C^2(\overline{\Omega}_0)}$ such that $\|u\|_{\delta,(0,T)\times\Omega_1} \le L_1$.

Proof: A hypothesis of Proposition 1 guarantees the result for $\delta = 1$. Suppose $1 < \delta < \frac{n+2}{n+1}$. Then $p = \frac{\delta}{\delta-1} > n+2$. Thus, if we choose $\tau = 0$, $s = T$, $a = 1$ and $q = 1$ in Lemma 3.2, then Lemma 3.1 guarantees the hypotheses of Lemma 3.2. That is,

$$
\begin{aligned}
\|u\|_{\delta,(0,T)\times\Omega_1} \le{}& L[\|u(0,\cdot)\|_{1,\Omega_0}+\|u\|_{1,(0,T)\times\Omega_0}\\
&+\|b\|_{\infty,(0,T)\times(\partial\Omega''\cap\partial\Omega)}+\|M_2\|_{\infty,(0,T)}].
\end{aligned}
$$

The result follows. ■

Lemma 3.4: Suppose $0 < T < \infty$. Let $\frac{n+2}{n+1} \le \delta < \frac{n+2}{n}$. Then there exists $L_2 > 0$ dependent upon δ and $\|g_2\|_{C^2(\overline{\Omega}_0)}$ such that $\|u\|_{\delta,(0,T)\times\Omega_2} \le L_2$.

Proof: Suppose $\frac{n+2}{n+1} < \delta < \frac{n+2}{n}$. Set $p = \frac{\delta}{\delta-1}$. Then $\frac{n+2}{2} < p < n+2$. Also, if $a = \frac{p(n+2)}{(p-1)n+2)+p}$ then $\frac{p(n+2)}{n+2-p} = \frac{a}{a-1}$ and $1 < a < \frac{n+2}{n+1}$. Thus, if we choose $\tau = 0$,

$s = T$, $a = \frac{p(n+2)}{(p-1)(n+2)+p}$ and $q = 2$ in Lemma 3.2, then Lemma 3.1 guarantees the hypotheses of Lemma 3.2. That is,

$$\begin{aligned}\|u\|_{\delta,(0,T)\times\Omega_2} \leq L[&\|u(0,\cdot)\|_{a,\Omega_1} + \|u\|_{a,(0,T)\times\Omega_1} \\ &+ \|b\|_{\infty,(0,T)\times(\partial\Omega''\cap\partial\Omega)} + \|M_2\|_{\infty,(0,T)}].\end{aligned}$$

Consequently, the result follows from Lemma 3.3. ■

We are now in a position to prove Proposition 1.

Proof of Proposition 1: Let $k \in \mathbb{N}$. We claim that there exists $L_{k+1} > 0$ dependent upon k and $\|g_{k+1}\|_{C^2(\overline{\Omega}_0)}$ such that $\|u\|_{(\frac{n+1}{n})^k,(0,T)\times\Omega_{k+2}} \leq L_{k+1}$. The claim is true for $k = 1$ from Lemma 3.4. Inductively, suppose the claim holds for $k = h$ and consider $k = h+1$. Note that if $1 < p < \frac{n+2}{2}$ then $\frac{pn}{n+1-p} < \frac{p(n+2)}{n+2-p}$. Furthermore, note that if $a > \frac{n+2}{n+1}$ and p satisfies $\frac{pn}{n+1-p} = \frac{a}{a-1}$, then $1 < p < \frac{n+2}{2}$. Now, set $a = (\frac{n+1}{n})^h$. Suppose p satisfies $\frac{pn}{n+1-p} = \frac{a}{a-1}$. Then we easily verify that $\frac{p}{p-1} = (\frac{n+1}{n})^k$. Now choose $\tau = 0, s = T$ and $q = k+1$ in Lemma 3.2. Then Lemma 3.1 and the induction hypothesis guarantee the hypotheses of Lemma 3.2. Consequently, the result of Lemma 3.2, the induction hypothesis and Hölder's inequality give the claim for $k = h+1$. Induction gives the claim for all $k \in \mathbb{N}$. Proposition 1 follows. ■

We now consider the case $T = \infty$.

Proof of Proposition 2: Let $\{t_i\}_{i=1}^{\infty}$ be a sequence of real numbers such that $i-1 < t_i < i$ and $\|u(t_i,\cdot)\|_{1,\Omega_0} \leq \|u\|_{1,(i-1,i)\times\Omega_0}$ for all $i \in \mathbb{N}$. First, let $1 \leq \delta < \frac{n+2}{n+1}$. Then similarly to the proof of Lemma 3.3 we obtain

$$\begin{aligned}\|u\|_{\delta,(t_i,t_i+2)\times\Omega_1} \leq L[&\|u(t_i,\cdot)\|_{1,\Omega_0} + \|u\|_{1,(t_i,t_i+2)\times\Omega_0} \\ &+ \|b\|_{\infty,(t_i,t_i+2)\times(\partial\Omega''\cap\partial\Omega)} + \|M_2\|_{\infty,(t_it_i+2)}]\end{aligned}$$

for all $i \in \mathbb{N}$, where L depends upon δ and $\|g_1\|_{C^2(\overline{\Omega}_0)}$, and is independent of i. Thus, there exists $K > 0$ such that $\|u\|_{\delta,(t,t+1)\times\Omega_1} < K$ for all $t \geq 0$, and if $\|b(t,\cdot)\|_{\infty,\partial\Omega''\cap\partial\Omega}$, $M_2(t), \|u\|_{1,(t,t+1)\times\Omega''} \to 0$ as $t \to \infty$ then $\|u\|_{\delta,(t,t+1)\times\Omega_1} \to 0$ as $t \to \infty$. Now suppose $\frac{n+2}{n+1} < \delta < \frac{n+2}{n}$. Proceeding as in the proof of Lemma 3.4 we set $p = \frac{\delta}{\delta-1}$. Then $\frac{n+2}{2} < p < n+2$. Also, if $a = \frac{p(n+2)}{(p-1)(n+2)+p}$ then $\frac{p(n+2)}{n+2-p} = \frac{a}{a-1}$ and $1 < a < \frac{n+2}{n+1}$. Hence, from the above there exists a sequence of real numbers $\{t_i\}_{i=1}^{\infty}$ such that $i-1 < t_i < i$ and $\|u(t_i,\cdot)\|_{a,\Omega_1} \leq \|u\|_{a,(i-1,i)\times\Omega}$. Then similarly to the proof of Lemma 3.4 we obtain

$$\begin{aligned}\|u\|_{\delta,(t_i,t_i+2)\times\Omega_2} \leq L[&\|u(t_i,\cdot)\|_{a,\Omega_1} + \|u\|_{a,(t_i,t_i+2)\times\Omega_1} \\ &+ \|b\|_{\infty,(t_i,t_i+2)\times(\partial\Omega''\cap\partial\Omega)} + \|M_2\|_{\infty,(t_i,t_i+2)}]\end{aligned}$$

with L dependent upon δ and $\|g_2\|_{C^2(\overline{\Omega}_0)}$, and independent of i. Thus, there exists $K > 0$ such that for all $t \geq 0, \|u\|_{\delta,(t,t+1)\times\Omega_2} < K$ and if $\|b(t,\cdot)\|_{\infty,\partial\Omega''\cap\partial\Omega}, M_2(t)$, $\|u\|_{1,(t,t+1)\times\Omega''} \to 0$ as $t \to \infty$ then $\|u\|_{\delta,(t,t+1)\times\Omega_2} \to 0$ as $t \to \infty$. We can now proceed inductively as in the proof of Proposition 1, employing the time sequencing as above, to obtain our result. $\blacksquare$

4. Extensions of Propositions 1 and 2, Sup-Norm Results and Further Applications

We begin this section by commenting on the extension of Propositions 1 and 2 to the case where (S2) is replaced by a Neumann-type boundary inequality. Actually, (S2) could be replaced with any boundary restriction which could interface with a modification of (3.1) in the duality argument for Lemma 3.2. For example, if

$$\mathcal{L}_i = d_i\Delta \qquad \text{for all} \qquad i = 1, \dots, m \tag{4.1}$$

with $d_i > 0$ constant and (S2) is replaced by

(S2)$'$ If $\partial\Omega'\cap\partial\Omega = \emptyset$ then $\partial\Omega''\cap\partial\Omega = \emptyset$, and if $\partial\Omega'\cap\partial\Omega \neq \emptyset$ then for all $k \in \{1, \dots, m\}$ there exists $b_k \in C(\mathbb{R}_+ \times (\partial\Omega'' \cap \partial\Omega), \mathbb{R}_+)$ such that $\partial u_k/\partial\eta \leq b_k$ on $(0,T) \times (\partial\Omega'' \cap \partial\Omega)$.

then (3.1) can be modified as

$$\begin{cases} \phi_t = d_k\Delta\phi + \theta, & \tau < t < \tau + s,\ x \in \Omega_0 \\ \partial\phi/\partial\eta = 0, & \tau < t < \tau + s,\ x \in \partial\Omega_0 \\ \phi = 0, & t = \tau,\ x \in \Omega_0 \end{cases} \tag{4.2}$$

and we still obtain our results. We state this extension of Proposition 1 and 2 below.

Proposition 3: If $\mathcal{L}_i$ is given by (4.1) and (S2)$'$ replaces (S2) then the results of Propositions 1 and 2 still hold.

To obtain sup norm results for semilinear parabolic systems, we employ a standard interior estimate result for scalar equations. Let $\widetilde{\Omega} \subseteq \mathbb{R}^n$ be a bounded domain, lying locally on one side of its $C^{2+\alpha}$ boundary, such that Ω', Ω given in section 2 satisfy $\widetilde{\Omega} \subseteq \Omega$, $\operatorname{dist}(\Omega', \Omega\backslash\widetilde{\Omega}) > 0$, and $\partial\widetilde{\Omega} \cap \partial\Omega = \emptyset$ if $\partial\Omega' \cap \partial\Omega = \emptyset$. Let $0 \leq T_0 < T_1 < T_2$, $1 < q < \infty$, and $k \in \{1, \dots, m\}$ and suppose $\zeta \in W_q^{1,2}((T_0,T_2) \times \widetilde{\Omega})$ satisfies

$$\zeta_t = \mathcal{L}_k\zeta + g \qquad T_0 < t < T_2,\ x \in \widetilde{\Omega}. \tag{4.3}$$

Lemma 4.1: There exists a constant C independent of g and ζ such that

$$\|\zeta\|_{W_q^{1,2}((T_1,T_2)\times\Omega')} \leq C\Big(\|g\|_{q,(T_0,T_2)\times\widetilde{\Omega}} + \|\zeta\|_{q,(T_0,T_2)\times\widetilde{\Omega}} + \|\zeta\|_{W_q^{1-1/2q,2-1/q}((T_0,T_2)\times(\partial\widetilde{\Omega}\cap\partial\Omega))}\Big)$$

Furthermore, if $\|a_{ij}^k\|_{C^{1,1}(\mathbb{R}_+\times\overline{\widetilde{\Omega}})}, \|c_i^k\|_{C(\mathbb{R}_+\times\overline{\widetilde{\Omega}})} \le K$ then C can be chosen dependent only upon $\alpha, K, q, T_2 - T_1, T_1 - T_0, \widetilde{\Omega}$ and $\operatorname{dist}(\Omega', \Omega\backslash\widetilde{\Omega})$.

Proof of Lemma 4.1: Ladyženskaja et al. [11, page 355] comment that the first portion of this Lemma can be obtained analogously to their Theorem 10.1 page 351. This is indeed the case, and hence we omit the straightforward, tedious proof. For the second portion of the Lemma, the bounds on a_{ij}^k, c_i^k and the modulus of continuity of a_{ij}^k give the result. ■

Remark: If ζ satisfies (for example) $\partial\zeta/\partial\eta + \gamma\zeta = \beta$ on $(T_0, T_2) \times (\partial\widetilde{\Omega} \cap \partial\Omega)$ then the term $\|\zeta\|_{W_q^{1-1/2q,2-1/q}((T_0,T_2\times(\partial\widetilde{\Omega}\cap\partial\Omega))}$ can be replaced by $\|\beta\|_{W_q^{1-1/2q,2-1/q}((T_0,T_2)\times(\partial\widetilde{\Omega}\cap\partial\Omega))}$.

We now apply Propositions 1, 2, and 3 and Lemma 4.1 to obtain sup norm results for solutions of certain semilinear parabolic systems. We first consider an extension of (1.2), (1.3). Suppose Ω is bounded and v_1, v_2 satisfy

$$\text{(4.4)} \qquad \begin{cases} v_{1_t} = d_1\Delta v_1 + a_1 v_1 + b_1 v_2 - c(x) f(v_1, v_2), & t > 0,\ x \in \Omega \\ v_{2_t} = d_2\Delta v_2 + a_2 v_1 + b_2 v_2 + c(x) f(v_1, v_2), & t > 0,\ x \in \Omega \end{cases}$$

$$\text{(4.5)} \qquad v_1 = \beta_1, v_2 = \beta_2, \qquad t > 0,\ x \in \partial\Omega$$

$$\text{(4.6)} \qquad v_1 = v_{1_0}, v_2 = v_{2_0}, \qquad t = 0,\ x \in \Omega$$

where $d_1, d_2 > 0$; $b_1, a_2 \ge 0$; $a_1, b_2 \in \mathbb{R}$; $c \in C^\alpha(\overline{\Omega}, [-1, 1])$; f is locally Lipschitz and nonnegative with $f(0, z) = 0$ and $f(z, 0) = 0$ for $z \ge 0$; and $\beta_1, \beta_2, v_{1_0}, v_{2_0}$ are smooth, nonnegative and satisfy the usual compatability conditions. Then standard results guarantee that (4.4)–(4.6) has a unique classical, nonnegative, noncontinuable solution on $[0, T_{\max}) \times \overline{\Omega}$. We first show that we can obtain a priori bounds for this solution in $L^1((0, T) \times \Omega)$. If $\beta_1 \equiv \beta_2 \equiv 0$ then we easily obtain

$$\frac{d}{dt}\int_\Omega (v_1 + v_2)dx \le \int_\Omega [(a_1 + a_2)v_1 + (b_1 + b_2)v_2]dx.$$

Consequently, if $M = \|v_{1_0} + v_{2_0}\|_{1,\Omega}$ and $\gamma = \max\{(a_1 + a_2), (b_1 + b_2)\}$ then $\|v_1(t,\cdot) + v_2(t,\cdot)\|_{1,\Omega} \le Me^{\gamma t}$. In the case where β_1, β_2 are not identically zero, we can still obtain estimates. If we set $w(t, x) = \frac{1}{d_{\max}}\int_0^t (d_1 v_1(s, x) + d_2 v_2(s, x))ds$ where $d_{\max} = \max\{d_1, d_2\}$ then w satisfies

$$\text{(4.7)} \qquad \begin{cases} w_t \le d_{\max}\Delta w + v_{1_0} + v_{2_0} + \dfrac{d_{\max}}{\min\{d_1, d_2\}} w & \text{on } (0, T_{\max}) \times \Omega \\ w = \dfrac{1}{d_{\max}}\displaystyle\int_0^t (d_1\beta_1(s, x) + d_2\beta_2(s, x))ds & \text{on } (0, T_{\max}) \times \partial\Omega \\ w \equiv 0 \quad \text{on } \{0\} \times \Omega. \end{cases}$$

Hence, maximum principles yield a growth rate on w and consequently we obtain $L^1((0,T)\times\Omega)$ a priori bounds for v_1 and v_2. Now, let $\Omega_+ = \{x \mid c(x) > 0\}$ and suppose $\Omega_+ \neq \emptyset$. Then Ω_+ is open. Now let $\Omega' \subseteq \Omega_+$ be a domain such that $\mathrm{dist}(\Omega', \Omega\backslash\Omega_+) > 0$. Then clearly (4.4) satisfies a Total Summing Condition with respect to v_1, v_2 and Ω'. The following Propositions give sup norm estimates for v_1 and v_2 on Ω'. In order to obtain these estimates we need the following assumption:

(4.8) There exist $L, a, \varepsilon \geq 0$ such that $|f(y,z)| \leq L(y+z+\varepsilon)^a$ for all $y, z \geq 0$.

Proposition 4: If $T_{\max} < \infty$ then there exists $K > 0$ such that

$$\|v_1 + v_2\|_{\infty,(0,T_{\max})\times\Omega'} < K.$$

Proposition 5: Suppose $T_{\max} = \infty$ and $\beta_1 \equiv \beta_2 \equiv 0$. If $\max\{(a_1+a_2),(b_1+b_2)\} = 0$ then $\|v_1+v_2\|_{\infty,(0,\infty)\times\Omega'} < \infty$. If $\max\{(a_1+a_2),(b_1+b_2)\} < 0$ and $\varepsilon = 0$ then $\|v_1(t,\cdot)+v_2(t,\cdot)\|_{\infty,\Omega'} \to 0$ as $t \to \infty$.

Remark: One can show that the decay rate in Proposition 5 is actually exponential.

Proof of Proposition 4: Since $\mathrm{dist}(\Omega', \Omega\backslash\Omega_+) > 0$ there exists a domain $\widetilde{\Omega} \subseteq \Omega_+$ with a smooth boundary such that $\widetilde{\Omega}$ lies locally on one side of $\partial\widetilde{\Omega}$, $\mathrm{dist}(\widetilde{\Omega}, \Omega\backslash\Omega_+) > 0$, and $\mathrm{dist}(\Omega', \Omega\backslash\widetilde{\Omega}) > 0$. Consequently, we easily obtain that (4.4) satisfies a Total Summing Condition with respect to v_1, v_2 and $\widetilde{\Omega}$. Thus, from Proposition 1 and the L^1 estimates above, $\|v_1+v_2\|_{p,(0,T_{\max})\times\widetilde{\Omega}} < \infty$ for all $1 < p < \infty$. If we now apply Lemma 4.1 with $k = 1$, $\mathcal{L}_1 = d_1\Delta$, $\zeta = v_1$ and $g = a_1v_1 + b_1v_2 - cf(v_1,v_2)$ then we obtain $\|v_1\|_{W_q^{1,2}((\delta,T_{\max})\times\Omega')} < \infty$ where $0 < \delta < T_{\max}$ and $1 < q < \infty$. By choosing q sufficiently large and applying the Sobolev Imbedding theorem, we obtain a sup norm bound for v_1. The result for v_2 is similar. ■

Proof of Proposition 5: The result follows from the L^1 estimates obtained above and an argument similar to the proof of Proposition 4 given on time intervals $(T-\delta, T+1)$ as $T \to \infty$. ■

We remark that if $d_i\Delta$ is replaced by the more general $\mathcal{L}_i$ in (4.4) and $\beta_1 \equiv \beta_2 \equiv 0$ then we still obtain L^1 a priori estimates as above. However, if β_1 and β_2 are not identically zero then it is not clear whether a priori estimates can be obtained in general. In the special case of

$$\mathcal{L}_i = d_i(t,x)\Delta + \sum_{\ell=1}^{n} c_\ell^i(t,x)\frac{\partial}{\partial x_\ell},$$

it is shown in Fitzgibbon, Morgan and Waggoner [5] that $L^2((0,T)\times\Omega)$ a priori bounds can be obtained for a large class of systems containing (4.4)-(4.6). Consequently, in these cases, extended versions to Propositions 4 and 5 are possible.

We also remark that if we consider the given system (4.4)-(4.6) with (4.5) replaced by

$$\frac{\partial v_i}{\partial \eta}+\gamma_i v_i=\beta_i, \qquad t>0, x\in\partial\Omega, i=1,2 \tag{4.9}$$

with $\gamma_i\geq 0$ and smooth, then L^1 a priori bounds can be obtained as above, and analogous results to Proposition 4 and 5 can be given.

We now consider a large class of systems containing (4.4), (4.5), (4.6) and fitting naturally into the framework of section 2. Let Ω be possibly unbounded and for $i=1,\ldots,m$ let F_i: $\mathbb{R}_+\times\overline{\Omega}\times\mathbb{R}^m\to\mathbb{R}^m$ be continuous. Suppose that for all $i\in\{1,\ldots,m\}, M_i\subseteq\mathbb{R}$ is connected with $0\in M_i$, $M=M_1\times\cdots\times M_m$ and $v=(v_k)\in C^{1,2}((0,T)\times\overline{\Omega},M)\cap C([0,T)\times\overline{\Omega},M)$ satisfies

$$\begin{cases} v_{k_t}=\mathcal{L}_k v_k+F_k(\cdot,\cdot,v) & \text{on } (0,T)\times\Omega\\ v_k\equiv 0 & \text{on } (0,T)\times\partial\Omega. \end{cases} \tag{4.10}$$

In addition suppose that for all $i\in\{1,\ldots,m\}$ there exist $h_i\in C^2(M_i,\mathbb{R}_+)$ such that

(i) $h_i(z)=0$ iff $z=0$,

(ii) $h_i''(z)\geq 0$ for all $z\in M_i$,

and

(iii) there exist $L_1\in\mathbb{R}$ and $L_2\in L^1((0,T)\times\Omega)$ such that

$$\sum_{i=1}^m h_i'(z_i)F_i(t,x,z)\leq L_1\sum_{i=1}^m h_i(z_i)+L_2(t,x) \quad \text{for all} \quad (t,x,z)\in[0,T]\times\Omega\times M.$$

Such structures have been applied extensively in [3], [4], [5], [14] and [15]. The function $\sum\limits_{i=1}^m h_i(z_i)$ has been termed a separable generalized Lyapunov function for the vector field $F=(F_k)$. Typically, this function has the form $\sum\limits_{i=1}^m z_i$ (in which case $M_i\subseteq\mathbb{R}_+$ for all i) or $\sum\limits_{i=1}^m z_i^2$, but can have other forms as well; see Gröger [6], [7]. Also, there exists a class of reaction-diffusion systems modelling an autocatalytic reacton for which infinitely many structures of this form exist for the given reaction vector field [15].

Now, suppose that we multiply the k^{th} partial differential equation in (4.10) by $h_k'(v_k)$ and set $u_k=h_k(v_k)$, $f_k=h_k'(v_k)F_k(\cdot,\cdot,v)$. Then $u=(u_k)$ satisfies for each $k\in\{1,\ldots,m\}$

$$\begin{cases} u_{k_t}\leq\mathcal{L}_k u_k+f_k, & \text{on } (0,T)\times\Omega\\ u_k\equiv 0, & \text{on } (0,T)\times\partial\Omega\\ u_k\geq 0, & \text{on } (0,T)\times\Omega. \end{cases} \tag{4.11}$$

If we now have $u_k(0,\cdot) \in L^1(\Omega)$ and integration by parts is permissable then we obtain

$$\frac{d}{dt}\int_\Omega \sum_{k=1}^m u_k(t,x)dx \le L_1 \int_\Omega \sum_{k=1}^m u_k(t,x)dx + \int_\Omega L_2(t,x)dx.$$

Consequently, L^1 a priori bounds can be obtained for each u_k. Thus, if there exists a domain $\Omega' \subseteq \Omega$ for which (4.11) is Totally Summable with respect to u_k and Ω' then we can obtain $L^p((0,T)\times\widetilde{\Omega})$ estimates for each u_k where (4.11) is also Totally Summable with respect to u_k and $\widetilde{\Omega}$ and $\text{dist}(\Omega', \Omega\backslash\widetilde{\Omega}) > 0$. Returning to (4.10) we may apply Lemma 4.1 along with conditions such as

$$|F_k(\cdot,\cdot,v)| \le L_3\bigg(\sum_{i=1}^m h_i(v_i)\bigg)^a + L_4$$

and $h_i(v_i) = \alpha_i v_i^{\gamma_i}$ to obtain sup norm bounds for the unknowns $v_1,\ldots,v_m$ on Ω'. If the Total Summing Condition holds on any subdomain of Ω, then such results can be used to extend results in [14], [15] to unbounded domains. We remark that in this case condition (S3) is a special case of the intermediate sum condition given in [14], [15] for systems with $\mathcal{L}_i = d_i\Delta$ on bounded domains.

We should also say that nonhomogeneous as well as other types of boundary conditions can appear in (4.10) so long as a priori estimates can be obtained and these boundary conditions can interact with the operators $\mathcal{L}_k$ in the duality arguments from Section 3.

Finally, the L^1 estimates obtained above for (4.10) still hold if $\mathcal{L}_k$ is degenerate elliptic. Consequently, the results in this work can be used to obtain estimates for solutions on domains “away from” the degeneracies.

We conclude this section by demonstrating how our theory can be applied to obtain global existence for a one dimensional flame propagation model. Recently several authors (e.g. Larrouturou[12], Avrin [2]) have considered the model

$$(4.12)\qquad \begin{cases} \theta_t = \theta_{xx} + f(Y,\theta), & t>0,\ x\in\mathbb{R} \\ Y_t = aY_{xx} - f(Y,\theta), & t>0,\ x\in\mathbb{R} \\ \theta(t,-\infty)=0,\ \theta(t,\infty)=1, & t>0 \\ Y(t,-\infty)=1,\ Y(t,\infty)=0, & t>0 \\ \theta(0,x)=\theta_0(x),\ Y(0,x)=Y_0(x), & x\in\mathbb{R} \end{cases}$$

where $d>0$, f is nonnegative and locally Lipschitz, $f(0,z)=0$ and $|f(w,z)| \le L_1(w+z)^a$ for all $(w,z)\in\mathbb{R}_+^2$, and the initial data θ_0, Y_0 are bounded and nonnegative. Here θ and Y represent nondimensionalized temperature and mass fraction of the reactant respectively, and $a = L^{-1}$ where L is the Lewis number. We do not concern ourselves with the technicalities of local existence, but rather assume local existence of a nonnegative classical solution on a time interval $(0, T_{\max})$ and obtain bounds which can typically be used to obtain global existence. With this in mind we set $M = \|\theta_0 + Y_0\|_\infty,\ d_{\max} = \max\{1, a\}$ and $w(t,x) = \int_0^t (\theta(s,x) + aY(s,x))ds$ for all $(t,x) \in [0, T_{\max}) \times \mathbb{R}$. Then w satisfies

$$\begin{cases} w_t \leq d_{\max}\Delta w + M, & t > 0,\ x \in \mathbb{R} \\ w(t, -\infty) = at,\ w(t,\infty) = t, & t > 0 \\ w(0,x) \equiv 0, & x \in \mathbb{R}. \end{cases} \tag{4.13}$$

Hence, $w(t,x) \leq (M + d_{\max})t$ for all $(t,x) \in (0, T_{\max}) \times \mathbb{R}$. Now, let n be an integer. Then

$$\int_0^t \int_{n-3}^{n+3} (\theta + Y) dx ds \leq Kt$$

with K independent of n. Furthermore, (4.12) clearly satisfies (after reordering the equations) a total summing condition with respect to Y, θ and $(n-2, n+2)$ independent of n. Consequently, (assuming $T_{\max} < \infty$) we obtain

$$\|\theta\|_{p,(0,T_{\max})\times(n-2,n+2)} + \|Y\|_{p,(0,T_{\max})\times(n-2,n+2)} \leq K_p$$

with K_p independent of n from Proposition 1 for all $1 < p < \infty$. Therefore, if we use the polynomial growth restriction on f given above along with $\widetilde{\Omega} = (n-2, n+2)$ and $\Omega' = (n-1, n+1)$ in Lemma 4.1, then we can obtain

$$\|\theta\|_{W_q^{1,2}((\varepsilon,T_{\max})\times\Omega')} + \|Y\|_{W_q^{1,2}((\varepsilon,T_{\max})\times\Omega')} \leq \widetilde{K}_q$$

independent of n for all $1 < q < \infty$. By taking q sufficiently large we will be guaranteed sufficient bounds and smoothness to guarantee continuation of solutions.

References

1. N.D. Alikakos, "L_p Bounds for Solutions of Reacton Diffusion Equations", Comm. Part Diff. Eq., 4 (1979) 827-868.

2. J.D. Avrin, "Qualitative Theory for a Model of Laminar Flames with Arbitrary Nonnegative Initial Data", to appear in JDE.

3. W.E. Fitzgibbon and J. Morgan, "Existence of solutions for a Class of Weakly Coupled Semilinear Elliptic Systems", JDE, 77, No. 2 (1989) 351-368.

4. ______________, "Steady State Solutions for Certain Reaction-Diffusion Systems", JNA-TMA, (in press).

5. W.E. Fitzgibbon, J. Morgan and S.J. Waggoner, "Weakly Coupled Semilinear Parabolic Evolution Systems", Annali de Mathematica Pura ed Applicata, (in press).
6. K. Gröger, "On the Existence of Steady States of Certain Reaction Diffusion Equations", Arch. Rat. Mech. Anal. (1986) 297-306.
7. ______________, "Asymptotic Behavior of Solutions to a Class of Reaction Diffusion Equations", Math. Nachr., 112 (1983) 19-33.
8. A. Haraux and A. Youkana, "On a Result of K. Masuda Concerning Reaction-Diffusion Equations", Tohoku Math. J. 40 (1988) 159-163.
9. S. Hollis, R.H. Martin and M. Pierre, "Global Existence and Boundedness in Reaction Diffusion Systems", SIAM J. Math. Anal., Vol. 18 (1987) 744-761.
10. Y.I. Kanel', "Cauchy's Problem for Semilinear Parabolic Equations with Balance Conditions", Trans. Diff. Uray., 20 No. 10 (1984) 1753-1760.
11. O.A. Ladyženskaja, V.A. Solonnikov and N.N. Ural'ceva, *Linear and Quasilinear Equations of Parabolic Type*, Amer. Math. Soc., Providence, 1968.
12. B. Larrouturou, "The Equations of One-Dimensional Unsteady Flame Propagation: Existence and Uniqueness", SIAM J. Math. Anal., Vol. 19, No. 1 (1988) 32-59.
13. K. Masuda, "On the Global Existence and Asymptotic Behavior of Solutions of Reaction Diffusion Equations", Hokkaido Math. J., 12 (1982) 360-370.
14. J. Morgan, "Global Existence for Semilinear Parabolic Systems", SIAM J. Math. Anal., Vol. 20, No. 5 (1989) 1128-1144.
15. ______________, "Boundedness and Decay Results for Reaction-Diffusion Systems", SIAM J. Math. Anal., Vol. 21, No. 5 (1990) 1172-1189.