

# Cones, $k$ -Cycles of Linear Operators, and Problem B4 on the 1993 Putnam Competition

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**1. Introduction.** Problem B4 on the Fifty-Fourth William Lowell Putnam Mathematical Competition was as follows.

B4. *The function  $K(x, y)$  is positive and continuous for  $0 \leq x \leq 1$ ,  $0 \leq y \leq 1$ , and the functions  $f(x)$  and  $g(x)$  are positive and continuous for  $0 \leq x \leq 1$ . Suppose that for all  $x$ ,  $0 \leq x \leq 1$ ,*

$$\int_0^1 f(y)K(x, y)dy = g(x) \quad \text{and} \quad \int_0^1 g(y)K(x, y)dy = f(x).$$

*Show that  $f(x) = g(x)$  for  $0 \leq x \leq 1$ .*

In this article we will present a very simple “geometric” solution to this problem which, unlike the solution presented in [2], lends itself naturally to generalization. We believe that the process is a very interesting exercise in abstraction—discovering the essence of a specific problem and exploiting it to obtain a far more general result. In addition to solving Problem B4, the theorem that we will present also has the following result as a corollary.

*Let  $A$  be a real  $n \times n$  matrix with positive entries. If  $x$  is a vector with positive entries and  $A^k x = x$  for some  $k \in \mathbf{N}$ , then  $Ax = x$ .*

**2. Solving B4.** Let us first observe that the function  $K$  in Problem B4 is the “kernel” of a linear integral operator  $\mathcal{K}$  defined on the set of continuous functions  $\varphi$  on  $[0, 1]$  by

$$\mathcal{K} : \varphi \mapsto \mathcal{K}\varphi$$

where

$$[\mathcal{K}\varphi](x) = \int_0^1 K(x, y)\varphi(y)dy \quad \text{for } 0 \leq x \leq 1.$$

Thus we can restate the problem as:

*Show that if  $f$  and  $g$  are positive, continuous functions on  $[0, 1]$  with  $\mathcal{K}f = g$  and  $\mathcal{K}g = f$ , then  $f = g$ .*

Now, what are the important properties of the operator  $\mathcal{K}$ ? First of all  $\mathcal{K}$  is a linear operator, since

- $\mathcal{K}(f + g) = \mathcal{K}f + \mathcal{K}g$  for all continuous functions  $f, g$  on  $[0, 1]$ , and
- $\mathcal{K}(cf) = c\mathcal{K}f$  for all continuous functions  $f$  on  $[0, 1]$  and all real numbers  $c$ .

Also, since the kernel  $K$  is a positive function on  $[0, 1] \times [0, 1]$ , it is very easy to see that the operator  $\mathcal{K}$  maps nonnegative continuous functions to nonnegative continuous functions; that is, if  $\varphi$  is nonnegative and continuous on  $[0, 1]$ , then  $\mathcal{K}\varphi$  is nonnegative and continuous on  $[0, 1]$ . But in fact, a stronger statement is true, which is nearly as easy to see and left to the reader as

*Exercise 1:* Show that if  $\varphi$  is nonnegative, continuous, and not identically zero on  $[0, 1]$ , then  $\mathcal{K}\varphi$  is *positive* and continuous on  $[0, 1]$ .

We contend that this property and linearity are the only important properties of the operator  $\mathcal{K}$  at work in the problem.

Let us now proceed with our solution to Problem B4. Suppose that  $f$  and  $g$  are positive continuous functions on  $[0, 1]$  with  $\mathcal{K}f = g$  and  $\mathcal{K}g = f$ . First observe that if  $f = \lambda g$  for some positive constant  $\lambda$ , then  $\mathcal{K}f = \lambda\mathcal{K}g$ , which implies that  $g = \lambda f$ . Consequently  $\lambda = 1$ , and we are done. So assume that  $f$  is not a constant multiple of  $g$ . Since  $f$  and  $g$  are each positive on  $[0, 1]$ , each of  $f - \lambda g$  and  $g - \lambda f$  will be positive on  $[0, 1]$  for all sufficiently small  $\lambda > 0$ . Also, for all sufficiently large  $\lambda > 0$ , neither  $f - \lambda g$  nor  $g - \lambda f$  is positive on  $[0, 1]$ . So let

$$\lambda^* = \sup \{ \lambda > 0 : f - \lambda g \text{ and } g - \lambda f \text{ are positive on } [0, 1] \}.$$

Then  $f - \lambda^*g$  and  $g - \lambda^*f$  are each nonnegative on  $[0, 1]$ , and at least one of these functions is not positive on  $[0, 1]$ . Suppose that  $f - \lambda^*g$  is not positive, and observe that

$$\mathcal{K}(g - \lambda^*f) = \mathcal{K}g - \lambda^*\mathcal{K}f = f - \lambda^*g.$$

This contradicts the fact that  $\mathcal{K}\varphi$  must be positive on  $[0, 1]$  whenever  $\varphi$  is nonnegative and continuous on  $[0, 1]$ . Thus we must conclude that  $f = g$ .

**3. Generalizing the Problem.** The framework in which we will work is that of a *normed linear space*. A *linear space* (or vector space) over the real numbers  $\mathbf{R}$  is essentially a set  $X$  on which operations of addition and scalar multiplication are defined in such a way that

$$x, y \in X \implies x + y \in X \quad \text{and} \quad x \in X, \alpha \in \mathbf{R} \implies \alpha x \in X.$$

Identity elements, additive inverses, and commutativity and associativity properties are necessary as well; we refer the reader to a book such as [1] or [3] for more details. A linear space  $X$  becomes a *normed linear space* when we define upon it a *norm*, i.e., a function  $\| \cdot \| : x \mapsto \|x\|$  from  $V$  into the nonnegative real numbers that provides a measure of magnitude, or distance to the zero element of  $X$ . For example,  $\mathbf{R}$  is a normed linear space with the usual definitions of addition and scalar multiplication and norm defined by absolute value:  $\|x\| = |x|$ . Also,  $\mathbf{R}^n$  is a normed linear space with the usual definitions of addition and scalar multiplication of vectors and norm defined by the Euclidean distance to the origin,  $\|x\| = \sum_{i=1}^n x_i^2$ . We refer the reader to [3, Chapter 10] for the list of properties that technically define a norm, but we mention that probably the most important of these is the triangle inequality:

$$\|x + y\| \leq \|x\| + \|y\|.$$

A norm on a linear space always defines a *metric*, or notion of distance, on that space by

$$\text{dist}(x, y) = \|x - y\|.$$

A sequence  $x_1, x_2, x_3, \dots$  in a normed linear space  $X$  *converges* to an element  $x_\infty$  of  $X$  if and only if  $\lim_{n \rightarrow \infty} \|x_n - x_\infty\| = 0$ . Given a subset  $\Omega$  of  $X$ , the *closure* of  $\Omega$ , denoted by  $\overline{\Omega}$ , is the set of all limits of convergent sequences in  $\Omega$ . If  $\overline{\Omega} = \Omega$ , we say that  $\Omega$  is *closed*. Thus  $\Omega$  is closed if and only if the limit of every convergent sequence in  $\Omega$  is contained in  $\Omega$ .

$\mathcal{C}[0, 1]$ , the set of continuous functions on  $[0, 1]$ , is an example of a normed linear space as well. The operations of addition and scalar multiplication are defined in the natural way that we are used to from calculus. The usual, and in a sense “best,” norm on  $\mathcal{C}[0, 1]$  is defined by

$$\|f\| = \max_{x \in [0, 1]} |f(x)|.$$

$\mathcal{C}_+[0, 1]$ , the set of nonnegative continuous functions on  $[0, 1]$ , is a subset of the normed linear space  $\mathcal{C}[0, 1]$ . It is not a subspace, however, because it is not closed under (negative) scalar multiplication. It is an example of what is called a *strict cone*. A subset  $C$  of any linear space is a *cone* if  $C$  is closed under addition and closed under multiplication by *nonnegative* scalars. A cone is *strict* if  $\varphi$  and  $-\varphi$  are in  $C$  only if  $\varphi = 0$ .

Any cone  $C$  induces a “partial ordering”  $\succeq$  on the entire space defined by

$$\varphi \succeq \psi \iff \varphi - \psi \in C.$$

In particular, note that this says that  $\varphi \in C \iff \varphi \succeq 0$ , where  $0$  represents the zero element of the space. The induced ordering is a *partial* ordering because, in general, it need not be the case that arbitrary elements  $x, y \in X$  satisfy either  $x \succeq y$  or  $y \succeq x$ .

In a *normed* linear space, we can talk about the *interior* of a cone  $C$ , which we will denote by  $\text{int } C$ , defined as the set of all  $x \in C$  for which some  $\varepsilon$ -neighborhood  $\mathcal{N}_\varepsilon = \{y \in X \mid \|x - y\| < \varepsilon\}$  of  $x$  is contained in  $C$ . In this case we write

$$\varphi \succ \psi \iff \varphi - \psi \in \text{int } C.$$

The reader should note that  $\{x \in \mathbb{R} \mid x \geq 0\}$  is an example of a strict cone in  $\mathbb{R}$  with interior  $\{x \in \mathbb{R} \mid x > 0\}$  and that  $\{x \in \mathbb{R}^n \mid x_i \geq 0 \forall i\}$  is an example of a strict cone in  $\mathbb{R}^n$  with interior  $\{x \in \mathbb{R}^n \mid x_i > 0 \forall i\}$ . These are the standard *nonnegative cones* in these spaces; many other cones are possible in  $\mathbb{R}$  and many other *strict cones* are possible in  $\mathbb{R}^n$ ,  $n \geq 2$ .

In the context of Problem B4, where the cone  $C$  of interest is the set of nonnegative continuous functions on  $[0, 1]$ , it is not difficult to show that  $\text{int } C$  is precisely the set of *positive* continuous functions on  $[0, 1]$  and that the partial ordering induced by  $C$  says that

$$\varphi \succeq \psi \iff \varphi(x) \geq \psi(x) \text{ for all } x \in [0, 1],$$

and

$$\varphi \succ \psi \iff \varphi(x) > \psi(x) \text{ for all } x \in [0, 1].$$

Thus, this “nonnegative cone” in the linear space of continuous functions induces a very natural partial ordering on that space.

*Exercise 2:* Suppose that  $n > 1$  and  $x_1, x_2, \dots, x_n$  are elements of a strict cone  $C$ . Show (by induction) that if  $x_1 + x_2 + \dots + x_n = 0$ , then  $x_1 = x_2 = \dots = x_n = 0$ .

With the framework described above in place, we can state as follows a general, abstract theorem that encompasses Problem B4. Its proof will be essentially identical to the solution to Problem B4 given above.

**THEOREM 1:** *Suppose that  $X$  is a normed linear space and that  $C$  is a closed, strict cone in  $X$  with nonempty interior. Suppose further that  $\mathcal{L}$  is a linear operator on  $X$  with the property that*

$$\mathcal{L} : C \setminus \{0\} \longrightarrow \text{int } C.$$

*If  $x, y \in C$  are such that  $\mathcal{L}x = y$  and  $\mathcal{L}y = x$ , then  $x = y$ .*

*Proof:* Suppose that  $x$  and  $y$  are members of  $C$  with  $\mathcal{L}x = y$  and  $\mathcal{L}y = x$ . First observe that either  $x = y = 0$  or each of  $x$  and  $y$  is in  $\text{int } C$ ; so assume that  $x, y \in \text{int } C$ . Next observe that if  $x = \lambda y$  for some positive constant  $\lambda$ , then  $\mathcal{L}x = \lambda \mathcal{L}y$ , which implies that  $y = \lambda x$ . Consequently  $\lambda = 1$ , and we are done. So assume that  $x$  is not a constant multiple of  $y$ . Since  $x$  and  $y$  are in  $\text{int } C$ ,

[A<sub>1</sub>] *each of  $x - \lambda y$  and  $y - \lambda x$  are in  $\text{int } C$  for all sufficiently small  $\lambda > 0$ ,*

and

[A<sub>2</sub>] *for all sufficiently large  $\lambda > 0$ , neither  $x - \lambda y$  nor  $y - \lambda x$  is in  $\text{int } C$ .*

So let

$$\lambda^* = \sup \{ \lambda > 0 : x - \lambda y \text{ and } y - \lambda x \text{ are in } \text{int } C \}.$$

Then

[A<sub>3</sub>]  *$x - \lambda^* y$  and  $y - \lambda^* x$  are in  $C \setminus \{0\}$ , and at least one of these is not in  $\text{int } C$ .*

So suppose that  $x - \lambda^* y \notin \text{int } C$ , and observe that

$$\mathcal{L}(y - \lambda^* x) = \mathcal{L}y - \lambda^* \mathcal{L}x = x - \lambda^* y$$

This contradicts the assumption that  $\mathcal{L} : C \setminus \{0\} \rightarrow \text{int } C$ . Therefore  $x = y$ . ■

*Exercise 3:* Give detailed proofs of each of assertions A<sub>1</sub>, A<sub>2</sub>, and A<sub>3</sub> in the proof of Theorem 1.

**4. Further generalization:  $k$ -cycles.** Let  $\mathcal{L}$  be a linear operator on a linear space  $X$ . By a  $k$ -cycle of  $\mathcal{L}$ ,  $k \geq 1$ , we mean a collection of distinct elements  $\{x_1, x_2, \dots, x_k\} \subset X$  such that

$$\mathcal{L}x_1 = x_2, \mathcal{L}x_2 = x_3, \dots, \mathcal{L}x_{k-1} = x_k, \mathcal{L}x_k = x_1.$$

Note that Theorem 1 says that the operator  $\mathcal{L}$  described there can have no 2-cycle in  $C$ . Also note that a 1-cycle  $\{x_1\}$  of any operator  $\mathcal{L}$  is just a *fixed-point* of  $\mathcal{L}$  and that a  $k$ -cycle of  $\mathcal{L}$  gives rise to a fixed point of the  $k$ -fold composition  $\mathcal{L}^k$  of  $\mathcal{L}$ .

Our purpose here is to show that Theorem 1 can be generalized as follows.

**THEOREM 2:** *Suppose that  $X$  is a normed linear space and that  $C$  is a closed, strict cone in  $X$  with nonempty interior. Suppose further that  $\mathcal{L}$  is a linear operator on  $X$  with the property that*

$$\mathcal{L} : C \setminus \{0\} \longrightarrow \text{int } C.$$

*Then  $\mathcal{L}$  can have no  $k$ -cycle in  $C$  with  $k > 1$ .*

The proof of Theorem 2 will rely upon the following lemma.

**LEMMA:** *Suppose that the hypotheses of Theorem 2 are satisfied and that  $\{x_1, x_2, \dots, x_k\}$  is a  $k$ -cycle of  $\mathcal{L}$  in  $C$  with  $k > 1$ . Then  $x_i - \lambda \sum_{j \neq i} x_j \neq 0$  for all  $i = 1, 2, \dots, k$  and all  $\lambda > 0$ .*

*Proof:* Assume the hypotheses stated in the Theorem 2, and suppose that  $\{x_1, x_2, \dots, x_k\}$  is a  $k$ -cycle of  $\mathcal{L}$  in  $C$  with  $k > 1$ . Note first that each  $x_i \in \text{int } C$ . Suppose for the sake of contradiction that  $1 \leq i \leq k$ ,  $\lambda > 0$ , and  $x_i - \lambda \sum_{j \neq i} x_j = 0$ . Without loss of generality, we can assume that  $i = 1$ . Now observe that in addition to

$$x_1 - \lambda \sum_{j \neq 1} x_j = 0,$$

we also have

$$\mathcal{L}\left(x_1 - \lambda \sum_{j \neq 1} x_j\right) = x_2 - \lambda \sum_{j \neq 2} x_j = 0.$$

Subtracting the second of these from the first shows that

$$(1 + \lambda)x_1 - (1 + \lambda)x_2 = 0,$$

from which we conclude that  $x_1 = x_2$ . This contradicts the assumption that  $\{x_1, x_2, \dots, x_k\}$  is a  $k$ -cycle of  $\mathcal{L}$ . ■

*Proof of Theorem 2:* Assume the hypotheses stated in the theorem, and suppose that  $\{x_1, x_2, \dots, x_k\}$  is a  $k$ -cycle of  $\mathcal{L}$  in  $C$  with  $k > 1$ . Note first that each  $x_i \in \text{int } C$ . It follows that, for all sufficiently small  $\lambda > 0$ , we have

$$x_i - \lambda \sum_{j \neq i} x_j \in \text{int } C \quad \text{for each } i = 1, 2, \dots, k.$$

Also, for all sufficiently large  $\lambda > 0$ , we have

$$x_i - \lambda \sum_{j \neq i} x_j \notin \text{int } C \quad \text{for each } i = 1, 2, \dots, k.$$

So let

$$\lambda^* = \sup \left\{ \lambda > 0 : x_i - \lambda \sum_{j \neq i} x_j \in \text{int } C \text{ for each } i = 1, 2, \dots, k \right\}.$$

By virtue of the preceding lemma, we have

$$x_i - \lambda^* \sum_{j \neq i} x_j \in C \setminus \{0\} \text{ for each } i = 1, 2, \dots, k.$$

Furthermore,  $x_i - \lambda^* \sum_{j \neq i} x_j \notin \text{int } C$  for some  $i = 1, 2, \dots, k$ , since the alternative would contradict the definition of  $\lambda^*$ . We can assume without loss of generality that  $i = 2$ ; that is,  $x_2 - \lambda^* \sum_{j \neq 2} x_j \notin \text{int } C$ . But  $x_1 - \lambda^* \sum_{j \neq 1} x_j \in C \setminus \{0\}$  and

$$\mathcal{L} \left( x_1 - \lambda^* \sum_{j \neq 1} x_j \right) = x_2 - \lambda^* \sum_{j \neq 2} x_j.$$

This contradicts the assumption that  $\mathcal{L} : C \setminus \{0\} \rightarrow \text{int } C$ . ■

Finally, we will prove the corollary stated in the Introduction.

**COROLLARY:** *Let  $A$  be a real  $n \times n$  matrix with positive entries. If  $v$  is a vector with positive entries and  $A^k v = v$  for some  $k \in \mathbf{N}$ , then  $Av = v$ .*

*Proof:* Let  $\mathcal{L}$  be the linear operator on  $\mathbb{R}^n$  defined by multiplication by  $A$ ; that is,

$$\mathcal{L}x = Ax \text{ for all } x \in \mathbb{R}^n.$$

Let  $C$  be the nonnegative cone in  $\mathbb{R}^n$ , that is,

$$C = \{(x_1, x_2, \dots, x_n) \mid x_i \geq 0, i = 1, 2, \dots, n\}.$$

Then

$$\text{int } C = \{(x_1, x_2, \dots, x_n) \mid x_i > 0, i = 1, 2, \dots, n\}.$$

Let  $v \in C \setminus \{0\}$ . Since  $A$  has positive entries,  $Av$  will have positive entries; that is,  $\mathcal{L}$  maps  $C \setminus \{0\}$  into  $\text{int } C$ . We can assume without loss of generality that  $k$  is the *least* positive integer for which  $A^k v = v$ . So the vectors  $v, Av, A^2v, \dots, A^{k-1}v$  form a  $k$ -cycle of  $\mathcal{L}$  if  $k > 1$ . According to Theorem 2, no such cycle can exist. Consequently  $k = 1$ ; that is,  $Av = v$ . ■

## References

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