

A Mathematica[®] Companion
for Differential Equations

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Contents

Preface

viii

0. An Introduction to *Mathematica*

| | |
|-----------------------------|----|
| 0.1 Getting Started | 1 |
| 0.2 Functions and Equations | 5 |
| 0.3 Lists and Rules | 13 |
| 0.4 Troubleshooting | 20 |

1. An Introduction to Differential Equations

| | |
|--------------------------------|----|
| 1.1 Verifying Solutions | 25 |
| 1.2 Using <code>DSolve</code> | 28 |
| 1.3 Plotting Solution Families | 30 |
| 1.4 Solutions by Integration | 32 |
| 1.5 Using <code>NDSolve</code> | 35 |
| 1.6 Implicit Solutions | 36 |

2. Linear First-Order Equations and Applications

| | |
|----------------------------------|----|
| 2.1 Linear First-Order Equations | 38 |
| 2.2 Baseball Trajectories | 40 |
| 2.3 Rocket Science | 45 |
| 2.4 Opening a Parachute | 50 |
| 2.5 RL Circuits | 54 |

3. Nonlinear First-Order Equations and Applications

| | |
|--|----|
| 3.1 Direction Fields, Isoclines, and Solution Curves | 58 |
| 3.2 <code>DEPlot</code> | 62 |
| 3.3 The Logistic Population Model | 65 |
| 3.4 The Spruce Budworm Problem | 73 |
| 3.5 A Half-Wave Rectifier Circuit | 76 |

4. Approximation of Solutions

| | |
|---|----|
| 4.1 Taylor Polynomials | 79 |
| 4.2 Picard Iteration | 81 |
| 4.3 Numerical Methods: Euler, Heun, and RK4 | 85 |
| 4.4 Derivation of Runge-Kutta Methods | 91 |

5. Coupled Pairs of First-Order Equations

| | |
|---|-----|
| 5.1 Direction Fields and the Phase Plane | 97 |
| 5.2 Predator-Prey Systems | 105 |
| 5.3 The Spruce Budworm Problem Revisited | 107 |
| 5.4 The Half-Wave Rectifier Circuit Revisited | 109 |
| 5.5 Chapter Projects | 111 |

6. Second-Order Equations and Applications

| | |
|---|-----|
| 6.1 Plotting Solutions | 114 |
| 6.2 The Simple Pendulum | 119 |
| 6.3 Linear Equations: The Green's Function | 122 |
| 6.4 Constant Coefficients: Beats, Resonance, and Frequency Response | 127 |
| 6.5 Laplace Transforms | 136 |
| 6.6 Series Solutions | 146 |

7. Linear Differential Systems

| | |
|--|-----|
| 7.1 Vectors and Matrices in <i>Mathematica</i> | 155 |
| 7.2 Linear Differential Systems | 159 |
| 7.3 Matrix Eigenvalues and Eigenvectors | 164 |
| 7.4 Autonomous Systems | 166 |
| 7.5 The Matrix Exponential | 172 |
| 7.6 Eigenvalues and Phase Portraits | 175 |

8. Nonlinear Autonomous Systems in the Plane

| | |
|--------------------------------------|-----|
| 8.1 Equilibrium Points and Stability | 181 |
| 8.2 Limit Cycles | 185 |
| 8.3 Hopf Bifurcation | 191 |
| 8.4 Chapter Projects | 194 |

9. A Gallery of Nonlinear Systems

| | |
|-----------------------------------|-----|
| 9.1 Competition | 196 |
| 9.2 Predator-Prey | 196 |
| 9.3 The Chemostat | 197 |
| 9.4 van der Pol's Equation | 197 |
| 9.5 Duffing's Equation | 197 |
| 9.6 The Fitzhugh-Nagumo Equations | 198 |
| 9.7 The Brusselator | 198 |
| 9.8 The Oregonator | 199 |
| 9.9 The Lorenz System | 199 |
| 9.10 The Rössler System | 200 |
| 9.11 The Scroll Circuit | 202 |
| 9.12 The Double Pendulum | 202 |
| 9.13 The Three-Body Problem | 203 |
| 9.14 Chapter Projects | 204 |

10. Fourier Series and Sturm-Liouville Eigenvalue Problems

| | |
|--|-----|
| 10.1 Fourier Series | 206 |
| 10.2 Sturm-Liouville Eigenvalue Problems | 215 |
| 10.3 Eigenfunction Expansions | 220 |

11. Linear Partial Differential Equations

| | |
|--------------------------------|-----|
| 11.1 The 1-D Diffusion Problem | 227 |
| 11.2 The 2-D Laplace Equation | 234 |
| 11.3 The 2-D Diffusion Problem | 241 |
| 11.4 The 1-D Wave Equation | 249 |
| 11.5 The 2-D Wave Equation | 255 |

| | |
|---------------------------------------|-----|
| Appendix. The DiffEqs Packages | 259 |
|---------------------------------------|-----|

| | |
|--------------|-----|
| Index | 265 |
|--------------|-----|

Preface

Mathematica[®] is a powerful and elegant tool with which one can explore essentially every branch of mathematics. But in particular, its ease of use and its extraordinary graphics capabilities make it an ideal learning laboratory for the elementary differential equations course. This manual is intended to provide students with a guide along their journey through this course.

The material here is organized along the lines of the standard, modern differential equations course—modern in the sense that numerical solutions, graphics, and systems are emphasized and introduced early, standard in the sense that exact solutions are given their deserved place when appropriate. With coverage of topics through nonlinear systems, Sturm-Liouville eigenvalue problems, and the diffusion and wave equations, this manual will be useful in both semesters of the typical two-semester sequence.

Several standard applications play an important role here: projectile motion with resistance, spring-mass systems, the simple pendulum, the logistic population model, and predator-prey and competition models. In addition, the spruce budworm problem and a half-wave rectifier circuit each appear in the context of both single equations and coupled systems. Chapter 9, A Gallery of Nonlinear Systems, contains numerous other applications, including the chemostat, the Fitzhugh-Nagumo equations, the Brusselator, the Oregonator, the double pendulum, and the three-body problem.

In numerous instances, we take advantage of *Mathematica*'s ability to animate graphics. In addition to animating graphs of solutions, we create “physical” animations of a simple pendulum and a spring-mass system. (Some of these animations, as well as dozens of others, can be found in the form of QuickTime[®] movies at www.prenhall.com/hollis.)

This entire manual was created with *Mathematica* 4.1. Each chapter is a *Mathematica* notebook.

Learning *Mathematica*

Even though most students called upon to use *Mathematica* in the differential equations course will have used *Mathematica* to some extent in calculus, we do not assume that the student has any prior experience with *Mathematica*. Yet we do assume that a student at the level of the differential equations course has sufficient technical savvy to pick up on the basic ideas fairly readily. The introductory chapter (Chapter 0) begins with a discussion of *how* one learns *Mathematica*, with special emphasis on using the wealth of information easily available in the Help Browser. This is followed by brief discussions of several of the more fundamental ideas in *Mathematica* related to functions, equations, lists, and rules. The chapter concludes with a section on troubleshooting, which is a distillation of years of experience helping students resolve difficulties (as well as solving our own problems) with *Mathematica*.

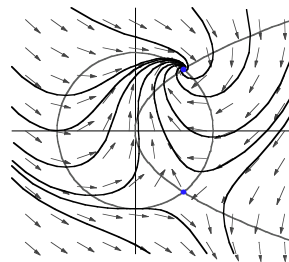
It is impossible to overstate the importance and usefulness of the Help Browser. The answer to virtually any question about *Mathematica* can be found there. For example, if one is unsure of the syntax of the `DSolve` function, then all one has to do is type and highlight “`DSolve`” and select Find Selected Function from the Help menu (or press the *help* key).

Throughout this manual we strive to present examples in a *style* of using *Mathematica* that is both transparent and efficient. We avoid the use of pure functions, for example, as well as most of the short “operator forms” such as `/@` for `Map`, etc., in order not to introduce unnecessary confusion or distract from our focus on differential equations. However, there will no doubt frequently arise questions about what some option is for or why a command is given a certain way. The way to find answers to such questions is to *change it and see what happens*. This cannot be emphasized enough: the user of this manual should actively experiment! Indeed, it can be argued that experimentation is precisely the purpose for which *Mathematica* exists.

The DiffEqs Packages

An important part of this manual is a collection of robust, fully documented, easy-to-install *Mathematica* packages, which can be found at www.prenhall.com/hollis. After an easy two-step installation process, the documentation will even appear in the Help Browser. An Appendix in this manual also provides documentation of these packages.

The `DEGraphics` package provides a number of functions that greatly simplify the process of plotting solutions of differential equations. In particular, `DEPlot` and `PhasePlot` make it easy to create very attractive plots of direction fields and solution curves with a single command. For example, the picture on the right was created with one command using `PhasePlot`. (For anyone who has used the standard *Mathematica* functions to create phase portraits, these routines are guaranteed to impress. To our knowledge, *no comparable package exists*.) Other functions in the package are `NDPlot`, `TimeStatePlot`, `ViewProjections`, `PlotImplicit`, and `PoincareTimeSection`.



The `DETools` package provides several functions that solve various types of equations encountered in the differential equations course. These include `FOLDSolve` (*First-Order-Linear-DSolve*), `BernoulliDSolve`, `RiccatiDSolve`, `SeparableDSolve`, and `SeriesDSolve`. Each of these returns solutions as expressions rather than rules, which makes them simpler for the novice to use than `DSolve`. `SeriesDSolve` is unique (as far as we know) in that it will automatically compute partial sums of Frobenius-series solutions as well as ordinary power-series solutions.

The `MatrixTools` and `FourierTools` packages provide several handy functions for working with matrices and Fourier series.

Final Thoughts

Writing ancillary such as this offers unique challenges to an author. Since the student will have available both a textbook and *The Mathematica Book* (as hard copy or through the Help Browser), one must avoid being pedantic about *both* of the things the manual is about: *Mathematica* and differential equations. So it is necessary to make many statements concerning both with little if any explanation. The key, I believe, is to give good examples and let the student take it from there. I hope I have achieved at least some success in that respect.

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