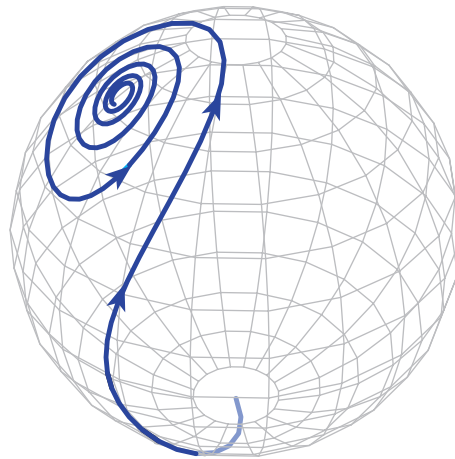


Differential Equations

with Boundary Value Problems

Selwyn Hollis

Armstrong Atlantic State University



PRENTICE HALL, Upper Saddle River, NJ 07458

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Preface

The introductory differential equations course plays an interesting role in the undergraduate mathematics curriculum. It is a required course for most science and engineering students, many of whom will take major courses that require certain knowledge of and skills related to differential equations. The course also plays the role of an introductory “applied course” for mathematics students, where modeling is often a primary focus.

Our point of view, which in no way diminishes the utility of the course’s content, is that the differential equations course has two overarching functions: (1) It is where students are introduced to the central subject in all of applied mathematics, a subject whose questions have spawned most of the classical theory of analysis since the time of Newton and continue to create a rich field of mathematical activity today. (2) It is a place to reinforce and extend the student’s (perhaps tenuous) understanding of calculus. It is the next step after calculus on the path that leads to real analysis. Whether the student will follow that path any farther makes little difference.

Throughout this book, we attempt to emphasize that the primary goal when studying a differential equation is always to understand the behavior of its solutions. Writing down formulas for solutions is just one means to that end. Qualitative and numerical methods are equally, if not more, important. Moreover, all of these “tools” are manifestations of theory, and we endeavor to emphasize that fact at a level which is appropriate for the course.

Times have changed. Amazingly fast computers are commonplace, and powerful software systems such as *Mathematica*, *Maple*, and MATLAB make it possible to solve highly complex problems and create wonderfully illuminating graphics. This computing power has enormous potential for enhancing the differential equations course—perhaps more so than in any other mathematics course. Indeed, the computer can facilitate the analysis of solutions algebraically, numerically, and graphically. Yet the computer cannot teach the conceptual and theoretical foundations that this book strives to convey—but neither can tedious algebraic manipulations done with paper and pencil. This book does not emphasize technology, and only occasionally do we mention it directly. However, we recommend the use of a computer to facilitate those computations that require more than a few routine algebraic steps. We are platform-neutral with respect to technology, and our belief is that the best way to address technology specifically is with separate, companion manuals. Thus we have created *A Mathematica Companion for Differential Equations* and *A Maple Companion for Differential Equations* for those who use *Mathematica* or *Maple* in their courses, and we highly recommend their use.

Organization and Content

For the most part, this book covers the traditional topics in the introductory differential equations course, while enhancing the usual algebraic approach with more geometric ideas and interpretations. Naturally reflecting the biases of the author (an inkling of which should be provided by the preceding commentary), certain standard topics are omitted or given less emphasis, while others are given increased emphasis. A few of the topics covered are rather novel for an introductory text. The following are some of our more significant deviations from the “standard course”:

- Systems and numerical methods are introduced early.
- Series methods are de-emphasized, particularly the method of Frobenius.
- The method of undetermined coefficients is banished to near-oblivion by the use of “exponential shift” and complex solutions.
- Linear equations of order three or higher are relegated to an appendix.
- An entire chapter is devoted to nonlinear systems in applications. We personally believe that this is where the subject of differential equations really becomes exciting and that students can greatly benefit from an introductory-level taste of some important models from biology, chemistry, and physics.

The book presupposes that the student has completed at least two, preferably three, semesters of calculus. Yet we realize that students at this stage have widely varying levels of mathematical maturity, and so we have taken some care to bring students along fairly slowly in the first few chapters. Overall, we have striven for a presentation that is neither overwhelming to weaker students nor patronizing to stronger ones.

We have aimed for a high degree of flexibility in terms of topic coverage. For instance, it is doubtful that all of the applications in Chapters 2 and 3 can be covered in any but the most applications-oriented course. Indeed, in the first four chapters, only Sections 1.3, 2.1, 3.1, 3.2, 4.2, 4.3 are truly essential. While the book as a whole is designed for a two-semester sequence, a single introductory course (or the first course in a sequence) can be taught in various ways, three of which are offered as follows:

- Thorough coverage of Chapters 1–6.
- Less thorough coverage of Chapters 1–7. (For instance, omit sections 2.2.3, 2.3, 3.5.3, 4.1, 4.5, and 4.6.)
- Minimal coverage of Chapters 1–4 followed by Chapters 8 and 9 plus topics from Chapter 10.

Linear algebra is often not a prerequisite for the differential equations course, though it should be. At some universities it is combined with differential equations to form a single course. Appendix I provides a reasonably complete development of elementary linear algebra, up to and including eigenvalues and

eigenvectors. This material can be touched upon or referred to as needed, or it can be wholly integrated into the course just prior to Chapter 8 (Linear First-Order Systems).

There are nearly 1500 exercises in the book. Appendix VI (Hints and Answers) contains hints for many of the less routine problems and answers to most of the odd-numbered problems. The *Instructor's Solutions Manual* contains solutions to all of the problems, and the *Student's Solutions Manual* contains solutions to all of the odd-numbered problems.

The website www.prenhall.com/hollis is the place to find *Mathematica* notebooks, *Maple* worksheets, animations, and general news and information, including the inevitable errata. Most importantly you will find applets for plotting direction fields and phase portraits, providing a convenient way for the reader to solve equations geometrically without access to a computer algebra system.

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Selwyn Hollis
shollis@armstrong.edu

↪ To the Student

This book is meant to be adaptable to a variety of teaching styles. While one professor may teach a traditional course that emphasizes pencil-and-paper solution techniques, another may leave the calculations almost entirely to a computer algebra system (*CAS*). We believe that students in a technology-based or technology-enhanced course should not be completely sheltered from computational details, especially those which cast light upon important concepts or reinforce the central ideas of calculus. So we have attempted to present calculations in a useful way without belaboring tedious or insignificant details. Occasionally, we skip to the end of a calculation that is long, laborious, and error-prone when done by hand, and we point out that the result was obtained by a *CAS*-aided computation. When reading this book, you should keep in mind your professor's particular emphasis with regard to this issue. It may be perfectly all right to gloss over certain details, accepting results more or less on faith and moving on, but even if not, never allow the trees to obscure your view of the forest.

Essentially the same advice applies with regard to proofs. We prove theorems whenever it is practical and instructive to do so. Often, however, a proof will require ideas from a more advanced course, and so we either omit the proof completely or attempt to explain the essence of it on an elementary level. In case you wish to investigate any of these “gaps” further—and we encouraged you to do so—a number of references are suggested in the bibliography (Appendix V). The bibliography also suggests sources where you can pursue various topics in greater depth.

The problem set at the end of each section usually begins with a number of computational problems of a routine nature, particularly in the early chapters. The purpose of these is to facilitate understanding of basic results, and they are designed to be done “by hand.” However, in a technology-based course, it may be acceptable for you to do these problems with a computer algebra system. Again, this is up to your professor. Other problems have you fill in certain omitted details in the section, direct you through the development of related concepts, or preview concepts that will be encountered later. Many of the problems will call upon your knowledge of calculus, which you will likely need to refresh from time to time with the help of the well-worn calculus book on your shelf.

The differential equations classroom is usually populated by students with diverse interests and goals. We hope that this book will be interesting, accessible, and relevant to each of you, regardless of your major. Differential equations are where calculus meets the real world—they are “calculus in action.” So enjoy the course. This should be the pay-off for the time and hard work you've invested in calculus.

