Test 3 Solutions

1. Consider the homogeneous equation \( y'' - \frac{1}{2t} y' + \frac{1}{2t^2} y = 0 \).

   a) Verify that \( u = t \) and \( v = \sqrt{t} \) are solutions on \((0, \infty)\).

   \[
   u'' - \frac{1}{2t} u' + \frac{1}{2t^2} u = 0 - \frac{1}{2t} 1 + \frac{1}{2t^2} t = 0
   \]

   \[
   v'' - \frac{1}{2t} u' + \frac{1}{2t^2} u = -\frac{1}{4t^{3/2}} - \frac{1}{2t} \frac{1}{2t^{1/2}} + t^{1/2} = 0
   \]

   b) Find the solution of \( y'' - \frac{1}{2t} y' + \frac{1}{2t^2} y = 0 \), \( y(1) = 1 \), \( y'(1) = 0 \).

   \[
   y(t) = c_1 t + c_2 \sqrt{t}
   \]

   \[
   y(1) = c_1 + c_2 = 1, \quad y'(1) = c_1 + \frac{1}{2} c_2
   \]

   \[
   c_1 = -1, \quad c_2 = 2
   \]

   \[
   y = -t + 2\sqrt{t}
   \]

   c) Construct the Green’s function.

   \[
   G(t, s) = \frac{s \sqrt{t} - t \sqrt{s}}{s \frac{1}{2\sqrt{s}} - \sqrt{s}} = \cdots = 2\sqrt{t} (\sqrt{t} - \sqrt{s})
   \]

   d) Use the Green’s function to compute the solution of \( y'' - \frac{1}{2t} y' + \frac{1}{2t^2} y = \sqrt{t} \), \( y(1) = 0 \), \( y'(1) = 0 \).

   \[
   \int_1^t 2\sqrt{t} (\sqrt{t} - \sqrt{s}) \sqrt{s} ds = 2\sqrt{t} \int_1^t (\sqrt{t}\sqrt{s} - s) ds
   \]

   \[
   = 2\sqrt{t} \left( \frac{2}{3} \sqrt{t} s^{3/2} - \frac{1}{2} s^2 \right) \bigg|_1^t
   \]

   \[
   = 2\sqrt{t} \left( \frac{2}{3} (t^2 - \sqrt{t}) - \frac{1}{2} (t^2 - 1) \right) = \frac{1}{3} \sqrt{t} (3 - 4\sqrt{t} + t^2)
   \]

2. Solve \( y'' + 7y' + 10y = 0 \), \( y(0) = 3 \), \( y'(0) = 0 \).

   \[
   r^2 + 7r + 10 = (r + 5)(r + 2)
   \]

   \[
   y(t) = c_1 e^{-5t} + c_2 e^{-2t}
   \]

   \[
   y(0) = c_1 + c_2 = 3
   \]

   \[
   y'(0) = -5c_1 - 2c_2 = 0 \quad \text{so} \quad c_1 = -2, \quad c_2 = 5
   \]

   \[
   y(t) = -2e^{-5t} + 5e^{-2t}
   \]
3. Solve \( y'' + 4y' + 13y = 0, \ y(0) = 0, \ y'(0) = 1. \)

\[
r^2 + 4r + 13 = 0 \quad \text{if} \quad r = \frac{1}{2}(-4 \pm \sqrt{-36}) = -2 \pm 3i.
\]

\[
y(t) = e^{-2t}(c_1 \cos 3t + c_2 \sin 3t)
\]

\[
y(0) = c_1 = 0, \quad y'(0) = c_2 e^{-2t}(-2 \sin 3t + 3 \cos 3t) \bigg|_{t=0} = 3c_2 = 1
\]

\[
y(t) = \frac{1}{3} e^{-2t} \sin 3t
\]

4. Solve \( y'' + 4y = 0, \ y(0) = 2, \ y'(0) = 1. \) Find the amplitude of the solution.

\[
y(t) = c_1 \cos 2t + c_2 \sin 2t
\]

\[
y(0) = c_1 = 2, \quad y'(0) = 2c_2 = 1
\]

\[
y(t) = 2 \cos 2t + \frac{1}{2} \sin 2t
\]

amplitude = \( \sqrt{4 + 1/4} = \sqrt{\frac{17}{4}} \)

5.a) Find a particular complex solution of \( z'' + 2z' + 10z = 10 e^{4it}. \)

\[
z = Ae^{4it}, \quad z' = 4i Ae^{4it}, \quad z'' = -16A e^{4it}
\]

\[
(-16 + 8i + 10)Ae^{4it} = 10e^{4it}
\]

\[
A = \frac{10}{-6 + 8i} = \frac{-5}{3 - 4i} = \frac{1}{5}(3 + 4i)
\]

\[
z = -\frac{1}{5}(3 + 4i)e^{4it}
\]

b) Find a particular real solution of \( y'' + 2y' + 10y = 10 \cos 4t. \)

\[
z = -\frac{1}{5}(3 + 4i)(\cos 4t + i \sin 4t) = -\frac{1}{5}\left(3 \cos 4t - 4 \sin 4t + i(3 \sin 4t + 4 \cos 4t)\right)
\]

\[
y = -\frac{1}{5}(3 \cos 4t - 4 \sin 4t)
\]

c) Find the amplitude \( A \) and the phase angle \( \phi \) so that the solution can be written in the form \( A \cos(\omega t - \phi) \).

\[
A = \frac{1}{5} \sqrt{3^2 + 4^2} = 1
\]

\[
\tan \phi = \frac{4}{3} \quad \text{(2nd quadrant)}
\]

\[
\phi = \tan^{-1}\left(-\frac{4}{3}\right) + \pi = \pi - \tan^{-1} \frac{4}{3}
\]

\[
y = \cos(4t - \phi)
\]

d) What is the gain of the system at the input frequency \( \frac{4}{2\pi} \) cycles/second?

\[
\frac{\text{response amplitude}}{\text{input amplitude}} = \frac{1}{1} = 1
\]