MATH 3411 Sample Problems for Test 1 with Solutions

1. A 100 gallon tank is initially full of pure water. Starting at time $t = 0$, salt water with a salt concentration of 1/10 kg/gallon flows into the tank at a rate of 5 gallons/minute. The fluid in the tank is kept well-mixed and flows out at the same rate. (a) Set up the initial-value problem for the amount $A(t)$ of salt in the tank at time t minutes, and (b) solve for $A(t)$.

$$\text{a)} \quad A' = 5 \times \frac{1}{10} - 5 \times \frac{A}{100}, \qquad A(0) = 0$$

$$\text{b)} \quad A' + \frac{1}{20}A = \frac{1}{2}$$

$$A = C\,e^{-t/20} + 10$$

$$0 = C + 10 \quad \Rightarrow \quad C = -10$$

$$A = 10(1 - e^{-t/20})$$

2. Consider the same scenario as in problem 1, *except* the fluid in the tank *flows out at a rate of* 10 gallons/minute. (a) Set up the initial-value problem for the amount $A(t)$ of salt in the tank for $0 \le t \le 20$ minutes, and (b) solve for $A(t)$. (*Hint*: The volume will be $V = 100 - 5t$.)

$$\text{a)} \quad A' = 5 \times \frac{1}{10} - 10 \times \frac{A}{100 - 5t}, \qquad A(0) = 0$$

$$\text{b)} \quad A' + \frac{2}{20 - t}A = \frac{1}{2}$$

$$\text{integrating factor: } e^{\int 2/(20-t)} = e^{-2\ln(20-t)} = \frac{1}{(20-t)^2}$$

$$\left(\frac{A}{(20-t)^2}\right)' = \frac{1}{2\,(20-t)^2}$$

$$\frac{A}{(20-t)^2} = \frac{1}{2\,(20-t)} + C$$

$$A = \frac{1}{2}(20-t) + C\,(20-t)^2$$

$$0 = 10 + 400C \quad \Rightarrow \quad C = -1/40$$

$$A = \frac{20-t}{2} - \frac{(20-t)^2}{40} = \frac{t\,(20-t)}{40}$$

3. Solve the initial-value problem

$$y' = 3t^2\,y^2, \quad y(0) = 1/8,$$

and state the maximal domain of the solution.

$$\frac{dy}{y^2} = 3t^2 dt \implies -\frac{1}{y} = t^3 + C \implies y = \frac{1}{C - t^3}$$

$$y(0) = 1/8 \implies C = 8 \implies y = \frac{1}{8 - t^3} \quad \text{for } -\infty < t < 2.$$

4. Use an integrating factor to find the general solution of $y' + \dfrac{\cos t}{1+\sin t}\, y = 1$.

$$\text{integrating factor: } e^{\frac{\cos t}{1+\sin t}} = e^{\ln(1+\sin t)} = 1+\sin t$$
$$((1+\sin t)\, y)' = 1+\sin t$$
$$((1+\sin t)\, y) = t - \cos t + C$$
$$y = \frac{t - \cos t + C}{1+\sin t}$$

5. Write down, *by inspection*, the general solution of each differential equation:

a) $y' + \frac{1}{2}\, y = 0$ b) $y' - 2y = 6$ c) $y' + 3t^2 y = 6t^2$

$y = C\, e^{-t/2}$ $y = C\, e^{2t} - 3$ $y = C\, e^{-t^3} + 2$

6. Find a solution of the equation $y\, y'' + 2(y')^2 = 0$ in the form t^a, where $a \neq 0$.

$$y = t^a \quad \Rightarrow \quad y' = a\, t^{a-1} \quad \Rightarrow \quad y'' = a\,(a-1)\, t^{a-2}$$
$$\begin{aligned} y\, y'' + 2(y')^2 &= t^a a\,(a-1)\, t^{a-2} + 2(a\, t^{a-1})^2 \\ &= (a^2 - a)\, t^{2a-2} + 2a^2 t^{2a-2} \\ &= (3a^2 - a)\, t^{2a-2} \\ &= a\,(3a-1)\, t^{2a-2} \\ &= 0 \ \text{ if } a = 0 \text{ or } a = 1/3 \end{aligned}$$
$$y = t^{1/3}$$

7. Given that $u = \cos 3t$ and $v = \sin 3t$ are solutions of $y'' + 9y = 0$ and that $w = e^{-t}$ is a particular solution of $y'' + 9y = 10e^{-t}$, find the solution of the initial-value problem

$$y'' + 9y = 10e^{-t}, \quad y(0) = 1, \ y'(0) = 0.$$

$$\text{general solution of the DE: } y = c_1 \cos 3t + c_2 \sin 3t + e^{-t}$$
$$\Longrightarrow \ y' = -3c_1 \sin 3t + 3c_2 \cos 3t - e^{-t}$$
$$y(0) = 1 \ \Longrightarrow \ 1 = c_1(1) + c_2(0) + 1 \ \Longrightarrow \ c_1 = 0$$
$$y'(0) = 0 \ \Longrightarrow \ 0 = -3c_1(0) + 3c_2(1) - 1 \ \Longrightarrow \ c_2 = 1/3$$
$$y = \frac{1}{3} \sin 3t + e^{-t}$$

8. A wood artifact contains 30% as much ^{14}C as it did when the material was living. Given that the half-life of ^{14}C is 5700 years, how old is the artifact?

$$A' = -k\, A, \quad A(0) = A_0$$
$$\Rightarrow \quad A(t) = A_0\, e^{-kt}$$
$$A(5700) = \frac{1}{2} A_0 \quad \Rightarrow \quad e^{-5700k} = \frac{1}{2} \quad \Rightarrow \quad k = \frac{\ln 2}{5700} \approx 0.000122$$
$$\Rightarrow \quad A(t) = A_0\, e^{-0.000122t}$$
$$.30A_0 = A_0\, e^{-0.000122t} \quad \Rightarrow \quad t \approx -\frac{\ln .3}{0.000122} \approx 9{,}870 \text{ years}$$

9. A can of soda is taken from the refrigerator with a temperature of 3° C. The temperature of the room is 23° C. If the temperature of the can of soda has risen to 8° C after 10 minutes, what will be its temperature after 30 minutes?

$$T' = -k\,(T - 23), \quad T(0) = 3$$
$$T(t) = 23 + C\,e^{-kt}, \quad T(0) = 3 \implies C = -20$$
$$T(t) = 23 - 20\,e^{-kt}$$
$$T(10) = 8 \implies e^{-10k} = 15/20 = 3/4 \implies k = -0.1\ln(3/4) \approx 0.0288$$
$$T(t) = 23 - 20\,e^{-.0288t}$$
$$T(3) = 23 - 20\,e^{-.0288(30)} \approx 14.6°$$

10. An object with mass $\frac{1}{2}$ kg is dropped from a height of 250 m and hits the ground 15 s later. (a) Find the object's linear drag coefficient k. (b) With what velocity does it hit the ground?

$$\frac{1}{2}v' + k\,v = -\frac{1}{2}(9.8), \quad v(0) = 0$$
$$v' + 2k\,v = -9.8, \quad v(0) = 0$$
$$\implies v = Ce^{-2kt} - \frac{4.9}{k}$$
$$v(0) = 0 \implies C = \frac{4.9}{k} \implies v = \frac{4.9}{k}\,(e^{-2kt} - 1)$$
$$\implies y = \frac{4.9}{k}\left(-\frac{1}{2k}e^{-2kt} - t\right) + C$$
$$y(0) = 250 \implies 250 = \frac{4.9}{k}\left(-\frac{1}{2k} - 0\right) + C \implies C = 250 + \frac{4.9}{2k^2}$$
$$\implies y = -\frac{4.9}{k}\left(\frac{1}{2k}e^{-2kt} + t\right) + 250 + \frac{4.9}{2k^2}$$
$$y(15) = 0 \implies -\frac{4.9}{k}\left(\frac{1}{2k}e^{-30k} + 15\right) + 250 + \frac{4.9}{2k^2} = 0$$
$$\implies k \approx 0.256 \quad \text{(from calculator solver)}$$

$$\text{(b)} \quad v = \frac{4.9}{k}\,(e^{-2kt} - 1) = 19.1\,(e^{-.51t} - 1)$$
$$\implies v(15) \approx -19 \text{ m/s}$$