1. A oil tank has the shape of a cylinder with its central axis horizontal. Its radius is 1 meter, and its length is 3 meters. The tank is initially full, and at time \( t = 0 \) oil begins to drain out of a hole at the bottom of the tank. The size of the hole and the viscosity of the oil are such that the constant \( \rho \) in Toricelli’s law is 0.003.

(a) Solve for the depth of the oil in the tank at time \( t > 0 \).

(b) At what time \( t \) does the tank become empty?

\[
\frac{6\sqrt{2y - y^2}}{\sqrt{y}} \, dy = -0.003 \, dt
\]

\[
\int_{y_{_{\text{in}}}}^{y} 6\sqrt{2 - u} \, du = -0.003 t
\]

\[
-4(2 - u)^{3/2} \bigg|_{y_{_{\text{in}}}}^{y_{\text{out}}} = -0.003 t
\]

\[
y = 2 - (0.00075 t)^{2/3}
\]

The tank is empty when

\[
t = \frac{1}{0.00075} 2^{3/2} \approx 3771 \text{ s} \quad (\text{a little over an hour})
\]

2. Find the first four terms of the Taylor-series expansion (i.e., up through the \( t^3 \)-term) of the solution of

\[
y' = e^{-t} - y^2, \quad y(0) = 1.
\]

\[
y(0) = 1
\]

\[
y'(0) = 1 - 1 = 0
\]

\[
y'' = -e^{-t} - 2yy', \quad \text{so} \quad y''(0) = -1 - 2(1)(0) = -1
\]

\[
y''' = e^{-t} - 2(y'y'' + yy''), \quad \text{so} \quad y'''(0) = 1 - 2(0 - 1) = 3
\]

Therefore,

\[
y = 1 - \frac{1}{2!} t^2 + \frac{3}{3!} t^3 + \cdots
\]

\[
= 1 - \frac{1}{2} t^2 + \frac{1}{2} t^3 + \cdots
\]

3. Describe three distinct global solutions of the initial-value problem

\[
y' = 3y^{2/3}, \quad y(0) = 0.
\]

By inspection, \( y = 0 \). Separating variables gives \( y = t^3 \). A third solution is \( y = \begin{cases} t^3, & \text{if } t \geq 0, \\ 0, & \text{if } t < 0. \end{cases} \)

\((-t^3\) is not a solution!)
4. Solve

\[ y' = 2y^{3/2}, \quad y(0) = \frac{1}{100} \]

and state the domain of the maximal solution.

\[ \frac{1}{2} \int_{100}^{y} u^{-3/2} du = \int_{0}^{t} ds \]

\[ -u^{-1/2}|_{100}^{y} = t \]

\[ 10 - y^{-1/2} = t \]

\[ y = \frac{1}{(10 - t)^2} \quad \text{for} \quad -\infty < t < 10 \]

5. Consider the equation \( y' = -y(y^2 - 30y + 200) \). Find the equilibrium solutions, and sketch a collection of representative solution curves (for \( t \geq 0 \)).

6. Consider a population governed by the logistic equation \( P' = \frac{1}{5} P \left( 1 - \frac{P}{100} \right) \). What constant harvesting rate \( R \) will result in an equilibrium population of 50?

\[ \frac{1}{5} 50 \left( 1 - \frac{50}{100} \right) - R = 0 \]

\[ R = 5 \]
7. Consider the system

\[
\frac{dx}{dt} = x^2 + y^2 - 1, \quad \frac{dy}{dt} = x^2 + y^2 - 4.
\]

Sketch the nullclines and the direction field of the system. Then sketch the solution curve corresponding to \(x(0) = 1, \ y(0) = -2\).

(I goofed on the initial point \((1, -2)\). It doesn’t give a very interesting curve. The curve shown below goes through \((0, 0)\).)