Being One's Own Experimental Unit in the Classroom

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Talk’s Main Points

Data collected in class motivates discussion of:

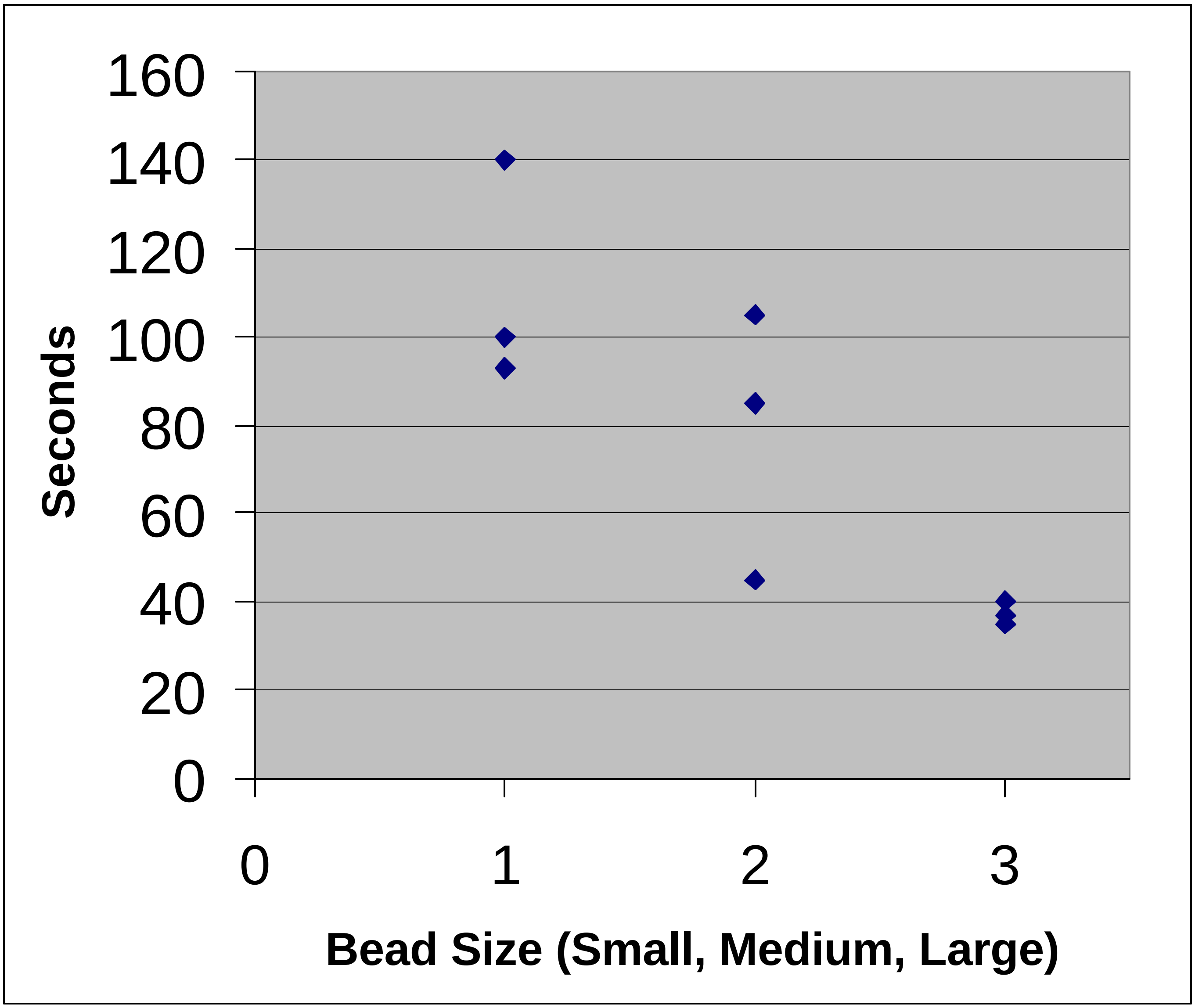
- ? Objectives or hypothesis
- ? Specifying a model of our outcome
- ? What assumptions are needed (violated?)
- ? Concepts of
  - ? Least Square Estimation
  - ? Normal Equations
  - ? Generalized Inverse
  - ? Estimable Function

Problem: Bead Jewelry Making

A manufacturer of bead rings can make any of 3 sizes of rings. Each contains 10 beads and materials costs of $1. Each can be sold for $5 and are equally demanded by the consumer public. Which product should she manufacture?
(Hint: how fast can they be made?)

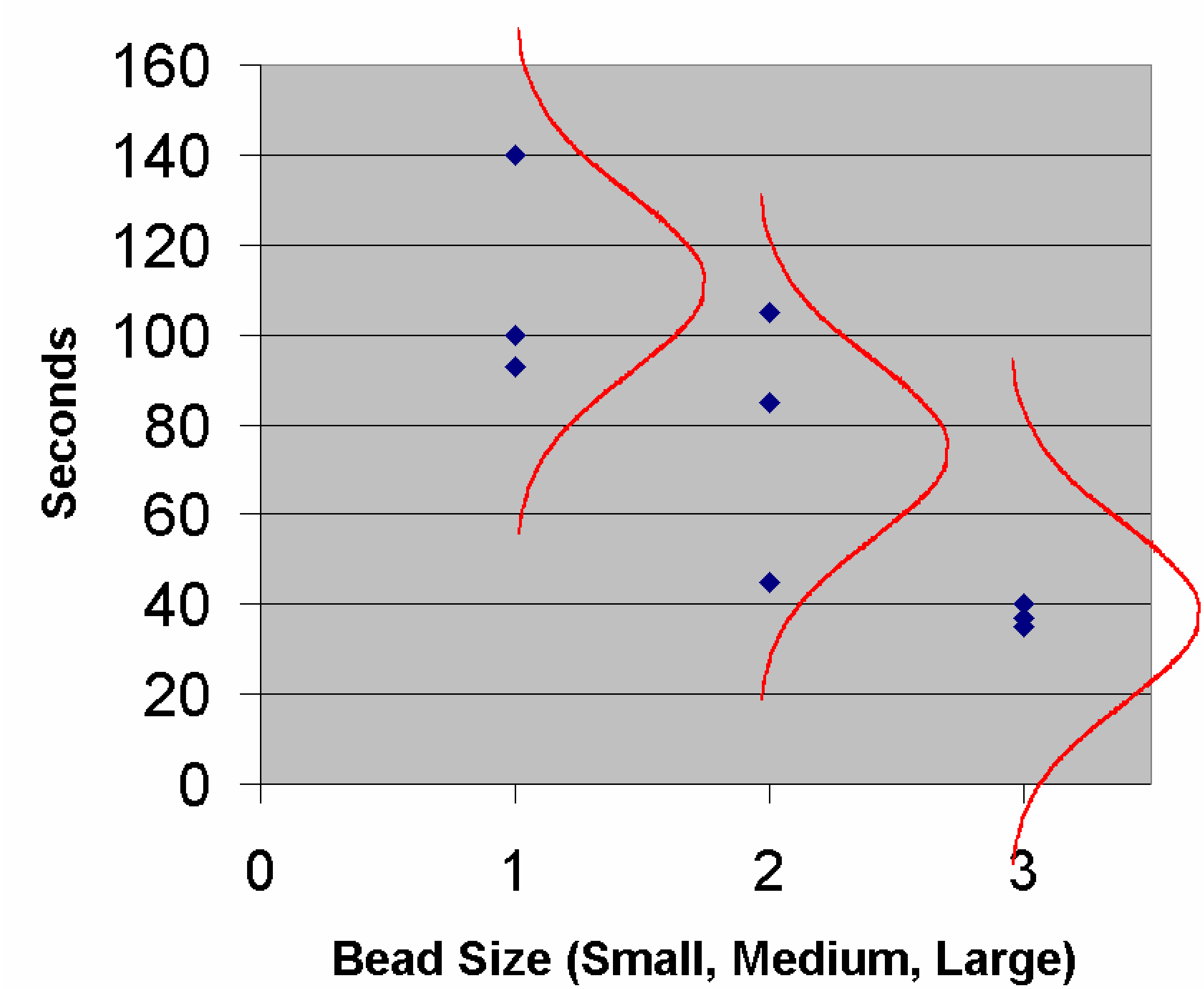
Students will suggest collecting a “pilot sample”.

beadsize	seconds
1(Small)	100
1(Small)	93
1(Small)	140
2(Medium)	105
2(Medium)	45
2(Medium)	85
3(Large)	35
3(Large)	40
3(Large)	37


160
140
120
100
80
60
40
20
0
Seconds
0
1
2
3
Bead Size (Small, Medium, Large)

Prompts these questions/comments (and others):

? Looks linear – regression?
? What is our objective, our hypothesis?
? Can we specify a model of our outcome?
? What assumptions are needed (violated?)


Seconds
160
140
120
100
80
60
40
20
0
0
1
2
3
Bead Size (Small, Medium, Large)

Looks linear – regression?

f(y;μ, σ)=

$$\frac{1}{\sqrt{2\Pi\sigma}}\exp\left[\left(-1/2\sigma^2\right)(y-\mu)^2\right],$$

$$-\infty<y<+\infty$$

?

?

?

where μ is a linear function of independent variable X (an indicator) and

σ is constant regardless of X

?

?

?

?

?

?

?

?

What assumptions are needed (violated?)

σ is constant regardless of X….not tenable, so

?

“Set aside the Large beads for a moment.”

What is our objective, our hypothesis?

Is there a difference in the stringing speed for small versus medium sized beads?

$\mu(\text{small}) = \beta_0 + \beta_1 X \quad X=0$

$\mu(\text{medium}) = \beta_0 + \beta_1 X \quad X=1$

or in the Experimental Design Vernacular

$\mu(\text{small}) = \mu + \tau_1$

$\mu(\text{medium}) = \mu + \tau_2$

where μ
could be a global mean and the τ's the treatment effect of each bead size

?

?

?

How To Estimate The Parameters

Think: Maximum Likelihood
Think: independent observations, i.e. product
Think: from a normal distribution

$$f(y_1, y_2, y_3, y_4, y_5, y_6; \mu, \tau_1, \tau_2, \sigma) =$$

$$\prod_{i=1}^{3} \frac{1}{\sqrt{2\Pi}\sigma} \exp\left[\left(-1/2\sigma^2\right)\left(y_i - (\mu + \tau_1)\right)^2\right] *$$

$$\prod_{i=4}^{6} \frac{1}{\sqrt{2\Pi}\sigma} \exp\left[\left(-1/2\sigma^2\right)\left(y_i - (\mu + \tau_2)\right)^2\right],$$

$$-\infty < y_1, y_2, y_3, y_4, y_5, y_6 < +\infty$$

or in matrix notation

$$\frac{1}{\sqrt{2\Pi\sigma^6}}\exp\left[\left(-1/2\sigma^6\right)\left(Y-X\beta\right)'\left(Y-X\beta\right)\right]$$
?

where

$$Y=\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{pmatrix}, \quad X=\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \beta=\begin{pmatrix} \mu \\ \tau_1 \\ \tau_2 \end{pmatrix}$$

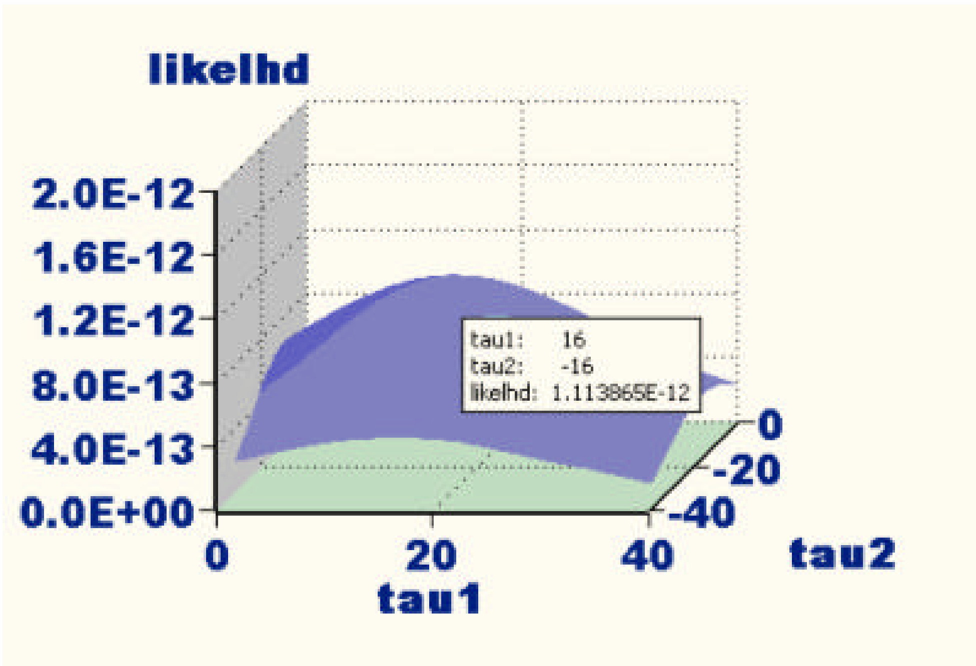
Treating the joint pdf as a function of the parameters and inquiring as to the likelihood of observing our collected speeds then we have:

$L(100, 93, 140, 105, 45, 85; \mu, \tau_1, \tau_2, \sigma)$

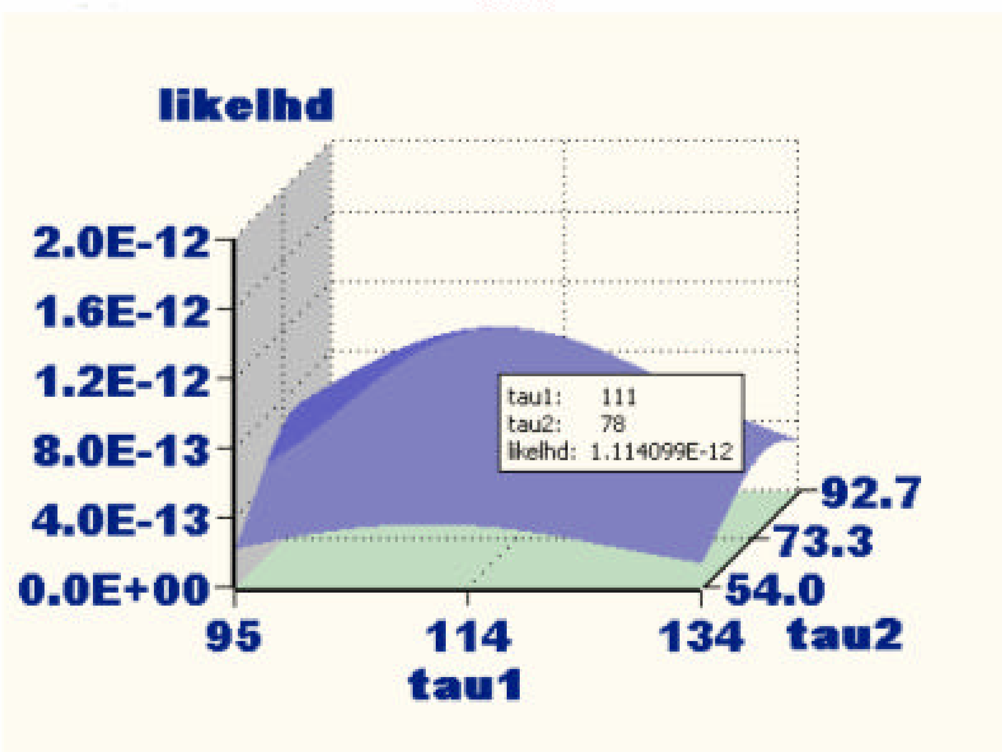
this likelihood function. What parameter values make this largest?

To view, let us set σ=28.07 seconds
And try μ=94.67 and μ=0
Allowing τ_1 and τ_2 to vary

Mu=94.67


likelhd
2.0E-12
1.6E-12
1.2E-12
8.0E-13
4.0E-13
0.0E+00
0
20
40
tau1
0
-20
-40
tau2
tau1: 16
tau2: -16
likelhd: 1.113865E-12

Mu=0


likelhd
2.0E-12
1.6E-12
1.2E-12
8.0E-13
4.0E-13
0.0E+00
95
114
134
tau1
92.7
73.3
54.0
tau2
tau1: 111
tau2: 78
likelhd: 1.114099E-12

Unsettling that our Likelihood is maximized in two (and actually infinite) places?

$$(\mu, \tau_1, \tau_2, \sigma) = (94.67, 16, -16, 28.07)$$

$$(\mu, \tau_1, \tau_2, \sigma) = (0, 111, 78, 28.07)$$

?

?

No unique maximum = no estimate for these parameters.

Allows introduction of the concepts of LEAST SQUARES ESTIMATION NORMAL EQUATIONS GENERALIZED INVERSE ESTIMABLE FUNCTION

By taking partial derivatives of L(.) or of logL(.) and setting to zero or noting that

$$\frac{1}{\sqrt{2\Pi}\sigma^6}\exp\left[\left(-1/2\sigma^6\right)\left(Y-X\beta\right)'\left(Y-X\beta\right)\right]$$

should maximize when

$$\left(Y-X\beta\right)'\left(Y-X\beta\right)$$

is minimized (LEAST SQUARES);

so solving $\left(Y-X\beta\right)=0$

(the NORMAL EQUATIONS)

$$X = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \text{so} \quad X'X = \begin{bmatrix} 6 & 3 & 3 \\ 3 & 3 & 0 \\ 3 & 0 & 3 \end{bmatrix}$$

and we have:

$$\begin{bmatrix} 6 & 3 & 3 \\ 3 & 3 & 0 \\ 3 & 0 & 3 \end{bmatrix} \begin{pmatrix} \mu \\ \tau_1 \\ \tau_2 \end{pmatrix} = X'Y = \begin{pmatrix} 6\overline{\overline{y}} \\ 3\overline{y}_1 \\ 3\overline{y}_2 \end{pmatrix} = \begin{pmatrix} 568 \\ 333 \\ 235 \end{pmatrix}$$

How To Solve?

$$(X'X)^g = \begin{bmatrix} 6 & 3 & 3 \\ 3 & 3 & 0 \\ 3 & 0 & 3 \end{bmatrix}^g = \begin{bmatrix} 1/3 & -1/3 & 0 \\ -1/3 & 2/3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Need a GENERALIZED INVERSE since Matrix is not of full rank.

?

$$\begin{pmatrix} \mu \\ \tau_1 \\ \tau_2 \end{pmatrix} =$$

$$(X'X)^g X'Y + \left[I - (X'X)^g X'X\right] b =$$

$$\begin{bmatrix} 1/3 & -1/3 & 0 \\ -1/3 & 2/3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 568 \\ 333 \\ 235 \end{pmatrix} + \begin{pmatrix} -b \\ b \\ b \end{pmatrix} = \begin{pmatrix} 78.33 - b \\ 32.67 + b \\ b \end{pmatrix}$$

This explains our multiple optima.

$$\begin{pmatrix} 78.33 - b \\ 32.67 + b \\ b \end{pmatrix}$$

<94.67, 16.33, -16.33> has b = -16.33

and

<0, 111, 78.33> has b = 78.33

So these parameters have no proper estimates…what can be estimated?

ESTIMABLE FUNCTIONS

"In the design model a linear function of the parameters is defined to be an estimable function if and only if there exists an unbiased estimator of it which is a linear function of the components of Y." Page 484, Theory and Application of the Linear Model, F.A. Graybill, 1976

ESTIMABLE FUNCTIONS

"For the one-way analysis of variance model, every estimable function is of the form

$$E\left[\sum_i \sum_i a_{it} y_{it}\right] = \sum_i b_i(\mu + \tau_i)$$

where

$$b_i = \sum_i a_{it}$$

and the a_{it} are real numbers" ; the i index indicates the group and the t index the observations within the group.

Page 37, <u>Design and Analysis of Experiments</u>, A. Dean and D. Voss, 1999

ESTIMABLE FUNCTIONS

“Are those functions of the design model parameters that have the same value at every possible optima”

View <94.67, 16.33, -16.33> and note that 16.33 – (-16.33) = 32.67

View <0, 111, 78.33> and note that 111 – 78.33 = 32.67

So, it appears that the difference in stringing speed between the treatment effect due to using small beads versus using medium beads can be estimated.

Now go on and talk about CONTRASTS….