1188. *Proposed by Javier Gomez-Calderon and David Wells, Penn State University at New Kensington.*

Find all real polynomials P having the property that $P(x-1)P(x) = P(x^2)$ for all x.

Solution by the **Armstrong Problem Solvers**, *Armstrong Atlantic State University, Savannah, GA.*

Either $P(x)$ is identically zero or $P(x) = (x^2+x+1)^n$ for some nonnegative integer n.

First notice that if $P(x) = (x^2+x+1)^n$ for some nonnegative integer n then $P(x-1) = (x^2-2x+1+x-1+1)^n = (x^2-x+1)^n$ and

$$\begin{aligned} P(x-1)P(x) &= (x^2-x+1)^n(x^2+x+1)^n \\ &= \left[(x^2+1)^2 - x^2\right]^n \\ &= (x^4+x^2+1)^n \\ &= P(x^2). \end{aligned}$$

Now suppose P is a real polynomial such that $P(x-1)P(x) = P(x^2)$ for all real numbers x. Evaluating at $x=1$ gives $P(0)P(1) = P(1)$, so either $P(1) = 0$ or $P(0) = 1$. If $P(1) = 0$, then we prove by induction that $P(2^{2^n}) = 0$ for all natural numbers n. Evaluating at $x = 2$ gives $0 = P(1)P(2) = P(4) = P(2^{2^1})$. If we assume that $P(2^{2^n}) = 0$ for some natural number n, then evaluating at $x = 2^{2^n}$ gives

$$0 = P(2^{2^n}-1)P(2^{2^n}) = P\left(\left(2^{2^n}\right)^2\right) = P(2^{2^{n+1}}).$$

Since P is a polynomial with an infinite number of zeros, P must be identically zero.

If $P(0) = 1$, then $P(-1)P(0) = P(0) = 1$, so $P(-1) = 1$. As above, if $P(1) = 0$, then $P(x) = 0$ for all real x, so $P(1) \neq 0$. Suppose α is a complex zero of P. Since $P(x-1)P(x) = P(x^2)$, then $P\left(\alpha^{2^n}\right) = 0$ for all positive integers n. Since P can have only finitely many zeros, α must be a kth root of unity for some positive integer k. Since $0 = P(\alpha)P(\alpha+1) = P\left((\alpha+1)^2\right)$, $(\alpha+1)^2$ must also be a root of unity, and

$$1 = |\alpha+1|^2 = |\alpha|^2 + 2\Re\{\alpha\} + 1,$$

where $\Re\{z\}$ denotes the real part of z. Thus $\Re\{\alpha\} = -\frac{1}{2}$, and α must be a third root of unity. Since the only complex zeros of $P(x)$ are third roots of unity other than 1, and $P(x)$ has real coefficients, then $P(x) = (x^2+x+1)^n$ for some nonnegative integer n.

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