1184. *Proposed by Arthur L. Holshouser, Charlotte, NC and Benjamin G. Klein, Davidson College, Davidson, NC*

Suppose that $(S, \odot_1)$ is a group with identity e_1. For s, arbitrary but fixed in S, define a binary operator on S by $a \odot_2 b = a \odot_1 s^{-1} \odot_1 b$ where a and b are elements of S and, for x in S, x^{-1} is the inverse of x in the group $(S, \odot_1)$. (a) Show that $(S, \odot_2)$ is a group and that $(S, \odot_2)$ is isomorphic to the group $(S, \odot_1)$. (b) Express $a \odot_1 b$ in terms of operations in the group $(S, \odot_2)$.

Solution by the **Armstrong Problem Solvers**, *Armstrong Atlantic State University, Savannah, GA.*

(a) Since $(S, \odot_1)$ is a group, S is closed under the operation $\odot_1$, and hence S is also closed under $\odot_2$. If a, b, and c are in S, then

$$(a \odot_2 b) \odot_2 c = (a \odot_1 s^{-1} \odot_1 b) \odot_1 s^{-1} \odot_1 c = a \odot_1 s^{-1} \odot_1 (b \odot_1 s^{-1} \odot_1 c) = a \odot_2 (b \odot_2 c)$$

and $\odot_2$ is associative.

If $a \in S$, then $a \odot_2 s = a \odot_1 s^{-1} \odot_1 s = a$ and $s \odot_2 a = s \odot_1 s^{-1} \odot_1 a = a$, so s is an identity element for $(S, \odot_2)$. In addition, $s \odot_1 a^{-1} \odot_1 s$ is an inverse for a in $(S, \odot_2)$ since

$$\begin{aligned} a \odot_2 (s \odot_1 a^{-1} \odot_1 s) &= a \odot_1 s^{-1} \odot_1 s \odot_1 a^{-1} \odot_1 s \\ &= a \odot_1 a^{-1} \odot_1 s \\ &= s \end{aligned}$$

and

$$\begin{aligned} (s \odot_1 a^{-1} \odot_1 s) \odot_2 a &= s \odot_1 a^{-1} \odot_1 s \odot_1 s^{-1} \odot_1 a \\ &= s \odot_1 a^{-1} \odot_1 a \\ &= s, \end{aligned}$$

so $(S, \odot_2)$ is a group.

Define a function $\phi : (S, \odot_1) \to (S, \odot_2)$ by $\phi(x) = x \odot_1 s$. Then, for any two elements a and b in S,

$$\begin{aligned} \phi(a) \odot_2 \phi(b) &= (a \odot_1 s) \odot_2 (b \odot_1 s) \\ &= a \odot_1 s \odot_1 s^{-1} \odot_1 b \odot_1 s \\ &= (a \odot_1 b) \odot_1 s \\ &= \phi(a \odot_1 b), \end{aligned}$$

so that ϕ is a homomorphism. If $\phi(a) = \phi(b)$, then $a \odot_1 s = b \odot_1 s$, so $a \odot_1 s \odot_1 s^{-1} = b \odot_1 s \odot_1 s^{-1}$, and $a = b$, so ϕ is injective. If $b \in S$, then $\phi(b \odot_1 s^{-1}) = b \odot_1 s^{-1} \odot_1 s = b$, so ϕ is surjective. Thus, ϕ is an isomorphism and $(S, \odot_2) \cong (S, \odot_1)$.

(b) If s^2 denotes $s \odot_1 s$, then $a \odot_1 b = a \odot_2 s^2 \odot_2 b$ since

$$\begin{aligned} a \odot_2 s^2 \odot_2 b &= (a \odot_1 s^{-1} \odot_1 s^2) \odot_2 b \\ &= (a \odot_1 s) \odot_2 b \\ &= a \odot_1 s \odot_1 s^{-1} \odot_1 b \\ &= a \odot_1 b. \end{aligned}$$

Armstrong Problem Solvers
Armstrong Atlantic State University
Department of Mathematics
11935 Abercorn Street
Savannah, GA 31419-1997
e-mail: James.Brawner@armstrong.edu