MM 1810. *Proposed by Greg Oman, Otterbein College, Westerville, OH.*
Let R be a ring. For elements $x, y \in R$ we say x divides y on the right if and only if there is a $z \in R$ with $xz = y$. (We denote this by $x|_r y$.) An element $p \in R$ is a right prime if and only if whenever $p|_r xy$, then either $p|_r x$ or $p|_r y$. Prove that if every element of R is right prime, then R is a division ring, that is, the nonzero elements of R form a group under multiplication. (Note: R is not assumed to be commutative nor is it assumed that R has a multiplicative identity.)

Solution by the **Armstrong Problem Solvers***, Armstrong Atlantic State University, Savannah, GA.*

Since R is a ring, $0 \in R$. If x and y are elements of R such that $xy = 0$, then $0 \cdot 0 = xy$ and $0|_r xy$. Since 0 is right prime, $0|_r x$ or $0|_r y$; thus $x = 0$ or $y = 0$. Notice that if $R = \{0\}$, then the nonzero elements of R do not form a group under multiplication, because there are no nonzero elements of R, so R is not a division ring in this case. For the remainder of the problem, we assume that R has at least one nonzero element, x. Since $x \cdot x = x^2$, $x|_r x^2$, and since x is right prime, then $x|_r x$, and $xe = x$ for some nonzero $e \in R$. We claim that e is a multiplicative identity in R.

Let $r \in R$. If $r = 0$, then $re = 0 = r = er$, so we assume that $r \neq 0$. Then $xer = xr$, so $x(er - r) = 0$, and since $x \neq 0$, then $er - r = 0$, and $er = r$ for all $r \in R$. Since $r \cdot r = r^2$, $r|_r r^2$, and since r is right prime, then $r|_r r$, and $rw = r$ for some nonzero $w \in R$. Since $er = r$ for all $r \in R$, then $ew = w$ and $rew = rw$, so $0 = rew - rw = (re - r)w$. Since $w \neq 0$, then $re - r = 0$ and $re = r$. Thus, $re = er = r$ for all $r \in R$, and e is a multiplicative identity in R. Notice that e is unique, since if v is another multiplicative identity in R, then $v = ve = e$.

Let n be any nonzero element of R. Then $n^2 e = n \cdot n$, so $n^2|_r n \cdot n$. Since n is right prime, then $n^2|_r n$ and $n^2 z = n$ for some nonzero $z \in R$. Thus, $n^2 z = ne$ and $n(nz - e) = 0$. Since $n \neq 0$, then $nz - e = 0$ and $nz = e$. In addition, $znz = ze = ez$, so $(zn - e)z = 0$, and since $z \neq 0$, then $zn - e = 0$ and $zn = e$. Thus, z is a multiplicative inverse for n. Since every nonzero element of R has a multiplicative inverse, then R is a division ring.

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