MM 1806. *Proposed by Michael Becker, University of South Carolina at Sumter, Sumter, SC.*

The intersection of the ellipsoid $x^2+y^2+\frac{z^2}{c^2}=1$ and the plane $x+y+cz=0$ is an ellipse. For $c>1$, find the value of c for which the area of the ellipse is maximal.

Solution by the **Armstrong Problem Solvers**, *Armstrong Atlantic State University, Savannah, GA.*

The area of the ellipse is maximal when $c=\sqrt{1+\sqrt{3}}$.

Since the ellipsoid is centered at the origin and the plane passes through the origin, the center of the ellipse is also at the origin. Thus, the maximum distance from the origin of any point on the ellipse will be the length of the semi-major axis, and the minimum will be the length of the semi-minor axis. We use the method of Lagrange multipliers to find these lengths by optimizing $f(x,y,z)=x^2+y^2+z^2$ subject to the constraints $g(x,y,z)=x^2+y^2+\frac{z^2}{c^2}=1$ and $h(x,y,z)=x+y+cz=0$. If (x,y,z) is a critical point of f, then $\nabla f=\lambda\nabla g+\mu\nabla h$, so

$$\begin{aligned} 2x &= 2\lambda x+\mu \\ 2y &= 2\lambda y+\mu \\ 2z &= \frac{2\lambda z}{c^2}+\mu c, \end{aligned}$$

and $\mu=2x(1-\lambda)=2y(1-\lambda)=\frac{2z}{c}\left(1-\frac{\lambda}{c^2}\right)$. Thus, either $x=y$ or $\lambda=1$.

If $x=y$ then $2x^2+\frac{z^2}{c^2}=1$ and $2x+cz=0$, so $z=-\frac{2x}{c}$. Thus, $x^2=\frac{c^4}{2(c^4+2)}=y^2$, $z^2=\frac{2c^2}{c^4+2}$, and $f(x,y,z)=\frac{c^4+2c^2}{c^4+2}$, which is greater than 1, since $c>1$.

If $\lambda=1$, then $\mu=0$, and $\frac{2z}{c}\left(1-\frac{1}{c^2}\right)=0$. Since $c>1$, we must have $z=0$, so $x^2+y^2=1$ and $x+y=0$, which gives $x^2=\frac{1}{2}=y^2$, and $f(x,y,z)=1/2+1/2=1$, which must be the absolute minimum, and $\frac{c^4+2c^2}{c^4+2}$ is the absolute maximum. Thus, the length of the semi-minor axis is 1, the length of the semi-major axis is $\sqrt{\frac{c^4+2c^2}{c^4+2}}$, and the area of the ellipse is $\pi\cdot\sqrt{\frac{c^4+2c^2}{c^4+2}}$.

To find the ellipse with maximal area, we maximize $f(c)=\frac{c^4+2c^2}{c^4+2}$ where $c>1$. We compute $f'(c)=\frac{-4c(c^4-2c^2-2)}{(c^4+2)^2}$, which is equal to zero when $c=\sqrt{1+\sqrt{3}}$. Since $f'(1)>0$ and $f'(2)<0$, f attains its absolute maximum at $c=\sqrt{1+\sqrt{3}}$. The maximal area is

$$\sqrt{f\left(\sqrt{1+\sqrt{3}}\right)}=\sqrt{\frac{1+\sqrt{3}}{2}}.$$

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