

MH 231, Double Progression. *Proposed by Mohammed K. Azarian, University of Evansville.* Let $a, b, B, c, C,$ and d be six positive real numbers such that a, b, c, d is an arithmetic progression and a, B, C, d is a geometric progression. Find all possible values of bc/BC .

Solution by the Armstrong Problem Solvers, Armstrong Atlantic State University, Savannah, GA.

The ratio bc/BC can be any real number greater than or equal to 1.

First we show that $bc/BC \geq 1$. Since a, b, c, d is an arithmetic progression, it must have a common difference, k , between consecutive terms. Thus, $b = a + k$, $c = a + 2k$, and $d = a + 3k$. Since a, B, C, d is a geometric progression, it must have a common ratio, r , between consecutive terms, so $B = ar$, $C = ar^2$, $d = ar^3$, and $BC = a^2r^3 = ad$. Thus,

$$\begin{aligned} \frac{bc}{BC} &= \frac{(a+k)(a+2k)}{a(a+3k)} \\ &= \frac{a^2 + 3ak + 2k^2}{a^2 + 3ak} \\ &= 1 + \frac{2k^2}{a(a+3k)}. \end{aligned}$$

Since a and $d = a + 3k$ are positive, then $\frac{2k^2}{a(a+3k)} \geq 0$ and $bc/BC \geq 1$. Notice that equality holds if and only if $a = b = c = d = B = C$.

Now, suppose p is any real number greater than or equal to 1. Let

$$k = \frac{3q + \sqrt{9q^2 + 8q}}{4},$$

where $q = p - 1 \geq 0$, and $r = \sqrt[3]{3k + 1} \geq 1$. Then, $a = 1$, $b = 1 + k$, $c = 1 + 2k$, $d = 1 + 3k$, $B = r$, and $C = r^2$ are all positive numbers, $BC = r \cdot r^2 = r^3 = 3k + 1$, $bc = (1 + k)(1 + 3k)$, and $bc/BC = 1 + q = p$.

Armstrong Problem Solvers
 Department of Mathematics
 Armstrong Atlantic State University
 Savannah, GA 31419-1997
 e-mail: James.Brawner@armstrong.edu