MH 226, Rational Trigonometric Sum Which rational angles α make $\sin(\alpha) + \cos(\alpha)$ a rational number?

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Using the sum-to-product formula $\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$, we see that

$$\begin{aligned} \cos\alpha + \sin\alpha &= \sin\left(\frac{\pi}{2} - \alpha\right) + \sin\alpha \\ &= 2\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4} - \alpha\right) \\ &= \sqrt{2}\cos\left(\frac{\pi}{4} - \alpha\right). \end{aligned}$$

From the article "Values of Trigonometric Functions," there is a small list of angles α for which $\sin(\alpha)+\cos(\alpha)$ is a rational number. Specifically, $\cos\left(\frac{\pi}{4} - \alpha\right)$ must equal $\pm\frac{\sqrt{2}}{2}$ or zero. Thus, either $\frac{\pi}{4} - \alpha = \frac{\pi}{4} + \frac{\pi}{2}k$ or $\frac{\pi}{4} - \alpha = \frac{\pi}{2} + \pi k$, where k is an integer. So, either $\alpha = \frac{\pi}{2}k$, in which case α is a quadrantal angle, or $\alpha = -\frac{\pi}{4} + \pi k$, where α is coterminal with either $\frac{3\pi}{4}$ or $\frac{7\pi}{4}$.

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