CMJ 894. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*
Consider ΔABC such that $A \leq 90°$. Let p, a, r, R, and h_a denote the semiperimeter, length of side $\overline{BC}$, inradius, circumradius, and altitude from A to $\overline{BC}$, respectively. Prove that

$$r(r+a) \leq pr \leq h_a R$$

with equality if and only if $A = 90°$.

Solution by the **Armstrong Problem Solvers**, *Armstrong Atlantic State University, Savannah, GA.*
Let D, E, and F be the points of tangency of the incircle with $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$, respectively, and let $b = AC$, $c = AB$, and $r' = AF$. Since the incenter I is the intersection of the angle bisectors of ΔABC, then $AE = AF = r'$, $BD = BF = c - r'$, $CD = CE = b - r'$, and $a = c - r' + b - r' = b + c - 2r'$. Since $m(\angle A) \leq 90°$, $m(\angle IAF) \leq 45° \leq m(\angle AIF)$, and $r \leq r'$, with equality if and only if $m(\angle A) = 90°$. Thus, $2r + 2a \leq 2r' + 2a = a + b + c = 2p$, and $r(r+a) \leq rp$.

Since $\overline{ID}$, $\overline{IE}$, and $\overline{IF}$ are perpendicular to the sides of ΔABC, the area of ΔABC is given by $rr' + r(c - r') + r(b - r') = r(b + c - r') = r(a + r') = rp$. Since $\overline{BC}$ is a chord of the circumcircle of ΔABC, $a \leq 2R$, with equality if and only if $\overline{BC}$ is a diameter of the circumcircle, which happens if and only if $\angle BAC$ is a right angle. Thus, the area of ΔABC is given by

$$rp = \frac{1}{2}h_a a \leq \frac{1}{2}h_a 2R = h_a R.$$

In each case, equality occurs if and only if $\angle BAC$ is a right angle.

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