

CMJ 889. Proposed by Jody M. Lockhart and William P. Wardlaw, U.S. Naval Academy, Annapolis, Maryland.

It is easy to see that the 3×3 matrix $\alpha = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ over the two-element field

\mathbf{F}_2 is a cyclic generator of the 8-element field \mathbf{F}_8 ; that is, $\mathbf{F}_8 = \{\mathbf{0}, \alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5, \alpha^6, \alpha^7\}$ consists of the zero matrix along with the powers of α . Determine how many 4×4 matrices over \mathbf{F}_8 have determinant $\alpha^2 + 1$.

Solution by the Armstrong Problem Solvers, Armstrong Atlantic State University, Savannah, GA.

There are $(8^4 - 1)(8^4 - 8)(8^4 - 8^2)(8^3) = 34,558,531,338,240$ such matrices. First notice that if A is a 4×4 matrix over \mathbf{F}_8 , then its determinant must be one of the 8 elements of \mathbf{F}_8 . Direct computation shows that α^7 is the unity in \mathbf{F}_8 and $\alpha^2 + 1 = \alpha^6$. For integers i with $1 \leq i \leq 7$, define M_i to be the set of 4×4 matrices over \mathbf{F}_8 with determinant equal to α^i . If $1 \leq i < j \leq 7$, then define a function $f : M_i \rightarrow M_j$ by $f(A) = A \cdot C$, where

$$C = \begin{bmatrix} \alpha^{j-i} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Notice that if $A \in M_i$, then $\det(f(A)) = \det(A) \cdot \det(C) = \alpha^i \cdot \alpha^{j-i} = \alpha^j \in M_j$. Since C is invertible with inverse

$$C^{-1} = \begin{bmatrix} \alpha^{7-j+i} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

then f is a bijection with inverse given by $f^{-1}(A) = A \cdot C^{-1}$, so M_i and M_j contain the same number of elements. It remains to count the number of 4×4 matrices over \mathbf{F}_8 with nonzero determinant, or equivalently, with linearly independent column vectors. There are $8^4 - 1$ choices for the first column, since it may be any vector other than the zero vector. The second column vector cannot be a multiple of the first, so there are $8^4 - 8$ choices for it. The third column cannot be a linear combination of the first two, so there are $8^4 - 8^2$ choices for it. Finally, the fourth column cannot be a linear combination of the first three, so there are $8^4 - 8^3$ for it, and a total of $m = (8^4 - 1)(8^4 - 8)(8^4 - 8^2)(8^4 - 8^3)$ matrices with nonzero determinant. Since each of the M_i has the same number of elements, each has $m/7 = (8^4 - 1)(8^4 - 8)(8^4 - 8^2)(8^3)$ elements.

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