CMJ 886. *Proposed by Árpád Bényi, Western Washington University, Bellingham, WA.*
Call a function f *good* if $f^{(2008)}(x) = -x$ for all $x \in \mathbf{R}$, where $f^{(2008)}$ denotes the function f composed with itself 2008 times. Prove the following:

(a) Every good function is bijective, odd, and non-monotonic.

(b) If f is good and $x_0 \neq 0$, there exist infinitely many 5-tuples $(p_1, p_2, p_3, p_4, p_5)$ of distinct positive integers whose sum is a multiple of 5 and for which, with $q_i := f^{(p_i)}(x_0)$, $q_1 \neq q_i$ for $i = 2, 3, 4, 5$, and $q_i \neq q_{i+1}$ for $i = 2, 3, 4$.

Solution by the **Armstrong Problem Solvers**, *Armstrong Atlantic State University, Savannah, GA.*
(a) Let f be a good function. If x and y are real numbers and $f(x) = f(y)$, then taking $f^{(2007)}$ of both sides gives $-x = -y$ and $x = y$, so f is injective. If $y \in \mathbf{R}$, then let $x = f^{(2007)}(-y)$, so $f(x) = f^{(2008)}(-y) = y$, and f is surjective, and hence, bijective.

To see that f is odd, note that $f\left(f^{(2008)}(x)\right) = f^{(2008)}(f(x))$, so that $f(-x) = -f(x)$ for all real numbers x.

We prove by induction that if f is monotonic, then $f^{(2n)}$ is increasing for all positive integers n. For the base case $n = 1$, let $a < b$ be real numbers. If f is increasing, then $f(a) < f(b)$ and $f^{(2)}(a) < f^{(2)}(b)$, so $f^{(2)}$ is increasing. If f is decreasing, then $f(a) > f(b)$ and $f^{(2)}(a) < f^{(2)}(b)$, so again, $f^{(2)}$ is increasing. For the inductive step, suppose that if f is monotone, then $f^{(2n)}$ is increasing. Let a and b be real numbers. Since $f^{(2n)}$ is increasing, $f^{(2n)}(a) < f^{(2n)}(b)$. If f is increasing, then $f^{(2n+1)}(a) < f^{(2n+1)}(b)$ and $f^{(2n+2)}(a) < f^{(2n+2)}(b)$, so $f^{(2(n+1))}$ is increasing. If f is decreasing, then $f^{(2n+1)}(a) > f^{(2n+1)}(b)$ and $f^{(2n+2)}(a) < f^{(2n+2)}(b)$, so again, $f^{(2(n+1))}$ is increasing. Thus, if f were monotonic, then $f^{(2(1004))}(x) = f^{(2008)}(x) = -x$ would be increasing, which is a contradiction.

(b) We claim that if n is any positive integer, then $(n, n+1, n+2, n+4, n+8)$ is such a 5-tuple. If $q_i = q_1$ for $i = 2$, 3, 4, or 5, then $f^{(p_i)}(x_0) = f^{(p_1)}(x_0)$, so $f^{(n+2^{i-2})}(x_0) = f^{(n)}(x_0)$. Since f is bijective, we can take $f^{(-n)}$ of both sides so that $f^{(2^{i-2})}(x_0) = x_0$, where 2^{i-2} is a divisor of 2008 for $i = 2$, 3, 4, and 5. Thus, $f^{(2008)}(x_0) = x_0 = -x_0$, so $x_0 = 0$, which is a contradiction. Similarly, if $q_{i+1} = q_i$ for $i = 2$, 3, or 4, then $f^{(p_{i+1})}(x_0) = f^{(p_i)}(x_0)$, and $f^{p_{i+1}-p_i}(x_0) = x_0$. Since $p_{i+1} - p_i = 1$, 2, or 4, this once again implies that $f^{(2008)}(x_0) = x_0 = -x_0$, and $x_0 = 0$. Finally, notice that $\sum_{i=1}^{5} p_i = 5(n+3)$, a multiple of 5.

Armstrong Problem Solvers
Armstrong Atlantic State University
Department of Mathematics
11935 Abercorn Street
Savannah, GA 31419-1997
e-mail: James.Brawner@armstrong.edu