AMM 11397. *Proposed by Grahame Bennett, Indiana University, Bloomington, IN.* Let a, b, c, x, y, z be positive numbers such that $a + b + c = x + y + z$ and $abc = xyz$. Show that if $\max\{x, y, z\} \geq \max\{a, b, c\}$, then $\min\{x, y, z\} \geq \min\{a, b, c\}$.

Solution by the Armstrong Problem Solvers, Armstrong Atlantic State University, Savannah, GA. Without loss of generality, assume $x \leq y \leq z$. If we let $f(t) = (t - x)(t - y)(t - z) = t^3 - (x + y + z)t^2 + (xy + yz + zx)t - xyz$ and $g(t) = (t - a)(t - b)(t - c) = t^3 - (a + b + c)t^2 + (ab + bc + ca)t - abc$, then the graph of $h(t) = f(t) - g(t) = (xy + yz + zx - ab - bc - ca)t = mt$ is a line L through the origin, with $h(a) = f(a)$, $h(b) = f(b)$, and $h(c) = f(c)$.

Suppose the slope m of the line L is positive. Then $g(z) = f(z) - h(z) = -mz < 0$. Since $g(t) \to \infty$ as $t \to \infty$, then $g(u)$ must be positive for some $u > z$. By the Intermediate Value Theorem, g must have a zero in the interval (z, u). Since the only zeros of g are a, b, and c, then $\max\{a, b, c\} > z = \max\{x, y, z\}$, which contradicts the hypothesis. If $m = 0$, then a, b, and c are zeros of f, so $\{x, y, z\} = \{a, b, c\}$, and hence the minima are equal. If $m < 0$, then $g(0) = -abc < 0$ and $g(x) = -h(x) = -mx > 0$, so by the Intermediate Value Theorem, g has a zero in the interval $(0, x)$. Since the zeros of g are a, b, and c, that means $\min\{a, b, c\} < x = \min\{x, y, z\}$.

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