AMM 11395. *Proposed by M. Farrokhi D.G., University of Tsukuba, Tsukuba, Japan.* Prove that if H is a finite subgroup of the group G of all continuous bijections of $[0,1]$ to itself, then the order of H is 1 or 2.

Solution by the Armstrong Problem Solvers, Armstrong Atlantic State University, Savannah, GA.

First notice that any element of G must be strictly monotone, since otherwise it would fail to be injective. Let G^+ denote the set of increasing members of G, and G^- the set of decreasing members of G.

Lemma 1: If $f, g \in G^-$, then $g \circ f \in G^+$.

Proof: If $0 \leq a < b \leq 1$, then $f(a) > f(b)$ and $g(f(a)) < g(f(b))$. $\diamond$

Lemma 2: If $g \in G^+$ has finite order, then g is the identity function.

Proof: If g is not the identity, than $g(a) \neq a$ for some $a \in [0,1]$. If $g(a) < a$, then since g is increasing, $(g \circ g)(a) < g(a) < a$, and $g^n(a) < a$ for all natural numbers n. Similarly, if $g(a) > a$, then $g^n(a) > a$ for all natural numbers n. Thus, if $g \in G^+$ has finite order, then g must be the identity function. $\diamond$

Lemma 3: If the order of $g \in G^-$ is finite, then the order of g is 2.

Proof: From Lemma 1, $g \circ g \in G^+$. Since the order of g is finite, then the order of $g \circ g$ is also finite, and by Lemma 2, $g \circ g$ is the identity function; thus, the order of g is 2. $\diamond$

Suppose H is a finite subgroup of G; then every element of H must have finite order. If $f \in H$ is increasing, then f must be the identity, by Lemma 2. If $f \in H$ is decreasing, then f must have order 2 by Lemma 3, so $f = f^{-1}$. If $f, g \in H$ are both decreasing, then $g \circ f$ is increasing by Lemma 1. Since $g \circ f \in G^+$ and has finite order, it must be the identity by Lemma 2, so $g = f^{-1} = f$. Therefore, H contains exactly one element of G^+ and at most element of G^-, so $|H| = 1$ or 2.

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