AMM 11395. Proposed by M. Farrokhi D.G., University of Tsukuba, Tsukuba, Japan. Prove that if \( H \) is a finite subgroup of the group \( G \) of all continuous bijections of \([0,1]\) to itself, then the order of \( H \) is 1 or 2.

Solution by the Armstrong Problem Solvers, Armstrong Atlantic State University, Savannah, GA.

First notice that any element of \( G \) must be strictly monotone, since otherwise it would fail to be injective. Let \( G^+ \) denote the set of increasing members of \( G \), and \( G^- \) the set of decreasing members of \( G \).

**Lemma 1:** If \( f, g \in G^- \), then \( g \circ f \in G^+ \).

**Proof:** If \( 0 \leq a < b \leq 1 \), then \( f(a) > f(b) \) and \( g(f(a)) < g(f(b)) \).

**Lemma 2:** If \( g \in G^+ \) has finite order, then \( g \) is the identity function.

**Proof:** If \( g \) is not the identity, than \( g(a) \neq a \) for some \( a \in [0,1] \). If \( g(a) < a \), then since \( g \) is increasing, \( (g \circ g)(a) < g(a) < a \), and \( g^n(a) < a \) for all natural numbers \( n \). Similarly, if \( g(a) > a \), then \( g^n(a) > a \) for all natural numbers \( n \). Thus, if \( g \in G^+ \) has finite order, then \( g \) must be the identity function.

**Lemma 3:** If the order of \( g \in G^- \) is finite, then the order of \( g \) is 2.

**Proof:** From Lemma 1, \( g \circ g \in G^+ \). Since the order of \( g \) is finite, then the order of \( g \circ g \) is also finite, and by Lemma 2, \( g \circ g \) is the identity function; thus, the order of \( g \) is 2.

Suppose \( H \) is a finite subgroup of \( G \); then every element of \( H \) must have finite order. If \( f \in H \) is increasing, then \( f \) must be the identity, by Lemma 2. If \( f \in H \) is decreasing, then \( f \) must have order 2 by Lemma 3, so \( f = f^{-1} \). If \( f, g \in H \) are both decreasing, then \( g \circ f \) is increasing by Lemma 1. Since \( g \circ f \in G^+ \) and has finite order, it must be the identity by Lemma 2, so \( g = f^{-1} = f \). Therefore, \( H \) contains exactly one element of \( G^+ \) and at most one element of \( G^- \), so \(|H| = 1\) or 2.

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