AMM 11394. *Proposed by K.S. Bhanu, Institute of Science, Nagpur, India, and M.N. Deshpande, Nagpur, India.* A fair coin is tossed n times, with $n \geq 2$. Let R be the resulting number of runs of the same face, and X the number of isolated heads. Show that the covariance of the random variables R and X is $n/8$.

Solution by the Armstrong Problem Solvers, Armstrong Atlantic State University, Savannah, GA.

For each positive integer n, let S_n denote the set of all possible sequences of n tosses. In addition, let $H_n = \{\alpha \in S_n : \alpha \text{ ends with H}\}$; let $T_n = \{\alpha \in S_n : \alpha \text{ ends with T}\}$, and more generally, let $\mathrm{HTT}_n = \{\alpha \in S_n : \alpha \text{ ends with HTT}\}$ for $n \geq 3$. Let a_n, b_n, and c_n denote the sum, over S_n, of R, X, and RX, respectively.

Notice that if a sequence of tosses has an additional toss appended, then the value of R will increase by 1 if the new toss has a different outcome than the previous toss; otherwise the value of R is unchanged. Thus,

$$\begin{aligned} a_{n+1} &= \sum_{\alpha\in H_n} [R(\alpha) + R(\alpha) + 1] + \sum_{\alpha\in T_n} [R(\alpha) + 1 + R(\alpha)] \\ &= 2\sum_{\alpha\in S_n} R(\alpha) + \sum_{\alpha\in S_n} 1 \\ &= 2a_n + 2^n. \end{aligned}$$

Since $a_1 = 2$ and $a_{n+1} = 2a_n + 2^n$ for $n \geq 1$, we show by induction that $a_n = (n+1)2^{n-1}$ for all positive integers n. If $n = 1$, then $(n+1)2^{n-1} = 2\cdot 2^0 = 2 = a_1$. If we assume that $a_n = (n+1)2^{n-1}$, then $a_{n+1} = 2\left((n+1)2^{n-1}\right) + 2^n = (n+2)2^n$.

If a sequence of tosses ends in HH, then the value of X will remain unchanged if an additional toss is appended. If a sequence of tosses ends in TH, then the value of X will decrease by 1 if an H is appended and will remain unchanged if a T is appended. If a sequence of tosses ends in T, then the value of X will increase by 1 if an H is appended and will remain unchanged if a T is appended. Thus, if $n \geq 2$, then

$$\begin{aligned} b_{n+1} &= \sum_{\alpha\in \mathrm{HH}_n} 2X(\alpha) + \sum_{\alpha\in \mathrm{TH}_n} [X(\alpha) - 1 + X(\alpha)] + \sum_{\alpha\in T_n} [X(\alpha) + 1 + X(\alpha)] \\ &= 2\sum_{\alpha\in S_n} X(\alpha) - \sum_{\alpha\in \mathrm{TH}_n} 1 + \sum_{\alpha\in T_n} 1 \\ &= 2b_n - 2^{n-2} + 2^{n-1} \\ &= 2b_n + 2^{n-2}. \end{aligned}$$

Since $b_2 = 2$ and $b_{n+1} = 2b_n + 2^{n-2}$ for $n \geq 2$, we show by induction that $b_n = (n+2)2^{n-3}$ for all integers $n \geq 2$. If $n = 2$, then $(n+2)2^{n-3} = 4 \cdot 2^{-1} = 2 = b_2$. If we assume that $n \geq 2$ and $b_n = (n+2)2^{n-3}$, then $b_{n+1} = 2\left[(n+2)2^{n-3}\right] + 2^{n-2} = (n+3)2^{n-2}$.

If $n \geq 2$, then

$$\begin{aligned} c_{n+1} &= \sum_{\alpha\in \mathrm{HH}_n} [R(\alpha)X(\alpha) + (R(\alpha)+1)\,X(\alpha)] \\ &\quad + \sum_{\alpha\in \mathrm{TH}_n} [R(\alpha)\,(X(\alpha)-1) + (R(\alpha)+1)\,X(\alpha)] \\ &\quad + \sum_{\alpha\in \mathrm{T}_n} [(R(\alpha)+1)\,(X(\alpha)+1) + R(\alpha)X(\alpha)] \end{aligned}$$

$$
\begin{aligned}
&= 2\sum_{\alpha\in S_n} R(\alpha)X(\alpha) + \sum_{\alpha\in S_n} X(\alpha) - \sum_{\alpha\in \mathrm{TH}_n} R(\alpha) + \sum_{\alpha\in T_n} R(\alpha) + 2\cdot 2^{n-2}\\
&= 2c_n + b_n + 2^{n-1} - \sum_{\alpha\in \mathrm{TH}_n} R(\alpha) + \sum_{\alpha\in \mathrm{HT}_n} R(\alpha) + \sum_{\alpha\in \mathrm{TT}_n} R(\alpha)\\
&= 2c_n + b_n + 2^{n-1} + \sum_{\alpha\in \mathrm{HTT}_n} R(\alpha) + \sum_{\alpha\in \mathrm{TTT}_n} R(\alpha)\\
&= 2c_n + b_n + 2^{n-1} + \sum_{\alpha\in \mathrm{H}_{n-2}} (R(\alpha)+1) + \sum_{\alpha\in \mathrm{T}_{n-2}} R(\alpha)\\
&= 2c_n + b_n + 2^{n-1} + \sum_{\alpha\in S_{n-2}} R(\alpha) + \sum_{\alpha\in \mathrm{H}_{n-2}} 1\\
&= 2c_n + b_n + 2^{n-1} + a_{n-2} + 2^{n-3}\\
&= 2c_n + b_n + a_{n-2} + 5\cdot 2^{n-3}\\
&= 2c_n + (n+2)2^{n-3} + (n-1)2^{n-3}\\
&= 2c_n + (n+3)2^{n-2}.
\end{aligned}
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Since $c_2 = 4$ and $c_{n+1} = 2c_n + (n+3)2^{n-2}$, we show by induction that $c_n = (n^2+5n+2)2^{n-4}$ for all integers $n \geq 2$. If $n = 2$, then $(n^2+5n+2)2^{n-4} = 16\cdot 2^{-2} = 4 = c_2$. If we assume that $n \geq 2$ and $c_n = (n^2+5n+2)2^{n-4}$, then

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\begin{aligned}
c_{n+1} &= 2\left[(n^2+5n+2)2^{n-4}\right] + (n+3)2^{n-2}\\
&= (n^2+5n+2+2n+6)2^{n-3}\\
&= (n^2+7n+8)2^{n-3}\\
&= \left[(n+1)^2+5(n+1)+2\right]2^{n-3}.
\end{aligned}
$$

Thus, with $n \geq 2$ tosses, the expected values for R, X, and RX are $a_n/2^n$, $b_n/2^n$, and $c_n/2^n$, respectively, and the covariance of R and X is

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\begin{aligned}
E(RX) - E(R)E(X) &= \frac{c_n}{2^n} - \frac{a_n}{2^n}\frac{b_n}{2^n}\\
&= \frac{(n^2+5n+2)2^{n-4}}{2^n} - \frac{(n+1)2^{n-1}}{2^n}\cdot\frac{(n+2)2^{n-3}}{2^n}\\
&= \frac{n^2+5n+2-(n+1)(n+2)}{2^4}\\
&= \frac{n^2+5n+2-(n^2+3n+2)}{16}\\
&= \frac{2n}{16}\\
&= \frac{n}{8}.
\end{aligned}
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