Problem 225. Help the ant find the path of shortest distance to the top of the icosahedron.

Solution by the Armstrong Problem Solvers, Armstrong Atlantic State University, Savannah, GA.

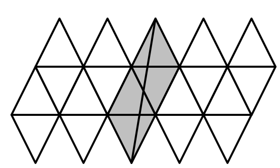
Since the faces of the icosahedron are flat, the ant's path of shortest distance will be the same distance as a straight-line path between corresponding points on a net of the icosahedron. An icosahedral net is shown below, with the ant's starting point corresponding to the lower vertex of any of the five triangles in the bottom row, and the top of the icosahedron corresponding to the top vertex of any of the five triangles in the top row. The shortest distance is then just the length of the diagonal of the shaded parallelogram shown below. If the equilateral triangles in the icosahedron have sides of length 1 unit, then this diagonal is the third side of a triangle with the other two sides of lengths $a = 1$ and $b = 2$ units, and an angle of C of 120 degrees. If c is the length of the third side, then the Law of Cosines tells us

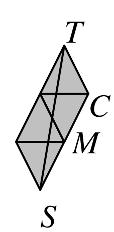
$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos C \\ &= 1 + 4 - 2\cdot 1\cdot 2\cos 120^\circ \\ &= 5 - 4(-\tfrac{1}{2}) \\ &= 7, \end{aligned}$$

so the shortest distance is $\sqrt{7} \approx 2.65$ units.

But since the ant is crawling on the icosahedron rather than its net, we still need to tell the ant which way to go! Let's call the ant's starting point S and say that the ant crosses edges of the equilateral triangles at points U, V, and W, in that order, before reaching the top, T. By symmetry, $SV = VT$. If C is the rightmost vertex of the shaded parallelogram, and M is the midpoint between S and C, then triangles SCW and SMU are similar. Since $SC = 2$ and $SM = 1$, then $SW = 2SU$, and $TW = SU$ by symmetry. Thus, SU is one third of ST, and U must be one third of the way from M across the top edge of the lowest shaded equilateral triangle. At last we can give the ant the following instructions:

> From the bottom vertex, crawl in a straight line along any of the lower five faces to a point one third of the way along the opposite edge of that face. On the next face, crawl in a straight line to the midpoint of the nearer of the other two edges in that face. Crawl up the next face to a point one third of the way along the horizontal edge of that face, and finally, crawl on the next face to the opposite vertex, which will be the top.





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