Putnam Seminar 2008, Problem Set 7 (Final Set)
Old Putnam Competition Problems

A1. Develope necessary and sufficient conditions which ensure that r_1, r_2, r_3 and r_1^2, r_2^2, r_3^2 are simultaneously roots of the equation $x^3 + ax^2 + bx + c = 0$.

A2. Let n be a given positive integer. How many solutions are there in ordered positive integer pairs (x, y) to the equation

$$\frac{xy}{x+y} = n?$$

A3. Assume that $f(x)$ is continuous in the interval $[0,1]$, prove that

$$\int_{x=0}^{x=1} \int_{y=x}^{y=1} \int_{z=x}^{z=y} f(x) f(y) f(z)\, dz\, dy\, dx = \frac{1}{3!} \left(\int_{t=0}^{t=1} f(t)\, dt \right)^3 .$$

A4. Let C be the unit circle $x^2 + y^2 = 1$. A point p is chosen randomly on the circumference of C and another point q is chosen randomly from the interior of C (these points are chosen independently and uniformly over their domains). Let R be the rectangle with sides parallel to the x-and$y$-axes with diagonal pq. What is the probability that no point of R lies outside of C?

A5. Consider the arithmetic progression

$$a,\ a+d,\ a+2d,\ \ldots$$

where a and d are positive integers. For any positive integer k, prove that the progression has either no exact kth powers or infinitely many.

A6. For each positive integer n, let $a(n)$ be the number of zeros in the base 3 representation of n. For which positive real numbers x does the series

$$\sum_{n=1}^{\infty} \frac{x^{a(n)}}{n^3}$$

converge?

B1. Prove or disprove: if x and y are real numbers with $y \geq 0$ and $y(y+1) \leq (x+1)^2$, then $y(y-1) \leq x^2$.

B2. A not uncommon calculus mistake is to believe that the product rule for derivatives says that $(fg)' = f'g'$. If $f(x) = e^{x^2}$, determine, with proof, whether there exists an open interval (a,b) and a nonzero function g defined on (a,b) such that this wrong product rule is true for x in (a,b).

B3. Let F be a field in which $1+1 \neq 0$. Show that the set of solutions to the equation $x^2+y^2=1$ with x and y in F is given by $(x,y)=(1,0)$ and

$$(x,y) = \left(\frac{r^2-1}{r^2+1}, \frac{2r}{r^2+1}\right),$$

where r runs through the elements of F such that $r^2 \neq -1$.

B4. Let k be the smallest positive integer with the following property:

There are distinct integers $m_1, \ldots, m_5$ such that the polynomial

$$p(x) = (x-m_1)\cdots(x-m_5)$$

has exactly k nonzero coefficients.

Find, with proof, a set of integers $m_1, \ldots, m_5$ for which this minimum k is achieved.

B5. For positive integers n, let $C(n)$ be the number of representations of n as a sum of nonincreasing powers of 2 where no power can be used more than three times. For example, $C(8)=5$ since the representations for 8 are:

$$8,\ 4+4,\ 4+2+2,\ 4+2+1+1, \text{ and } 2+2+2+1+1.$$

Prove or disprove that there is a polynomial $P(x)$ such that $C(n) = [P(n)]$ for all positive integers n; here $[u]$ denotes the greatest integers less than or equal to u.

B6. A "spherical ellipse" with foci A and B on a given sphere is defined as the set of all points P on the sphere such that $\widetilde{PA} + \widetilde{PB} =$ constant. Here $\widetilde{PA}$ denotes the shortest distance on the sphere between P and A. Determine the entire class of real spherical ellipses which are circles.