Putnam Seminar 2008, Problem Set 6
Topic: Geometry

Triangle Geometry

The **centroid** G of a triangle is the point where the three medians intersect. It is two-thirds of the way along the median from a vertex to the midpoint of the opposite side. The centroid is also the center of mass of a triangular plate with constant density.

The **orthocenter** H is the intersection of the three altitudes.

The **incenter** I is the intersection of the three angle bisectors, and is the center of the inscribed circle (incircle) of the triangle.

The **circumcenter** O is the intersection of the three perpendicular bisectors, and is the center of the circumscribed circle (circumcircle) of the triangle.

The points G, H, and O are collinear (they lie on the Euler line of the triangle), with G one-third of the way from O to H. The incenter does not lie on the Euler line unless the triangle is isosceles.

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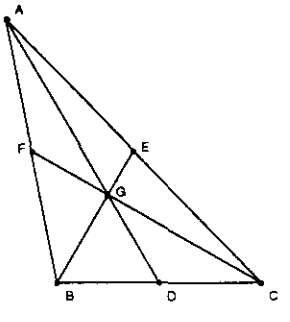
1. Let $A_1, A_2, \ldots, A_n$ be the successive vertices of a regular polygon inscribed in a circle of radius r and center O. Let P be a point on the ray $\overrightarrow{OA_1}$ extended beyond A_1. Show that

$$\prod_{i=1}^{n} PA_i = OP^n - r^n.$$

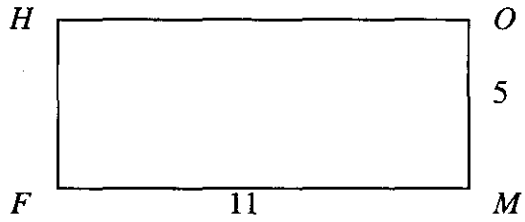
2. (i) Show that a regular hexagon, six squares, and six equilateral triangles can be assembled without overlapping to form a regular dodecagon.
(ii) Let $P_1, P_2, \ldots, P_n$ be the successive vertices of a regular dodecagon. Explain how the three diagonals $\overline{P_1P_9}$, $\overline{P_2P_{11}}$, and $\overline{P_4P_{12}}$ intersect.

3. In the figure below, D, E, and F are the midpoints of $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$, respectively, and G is the point of intersection of $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$. If ΔBDG is an equilateral triangle with edges of unit length, find the perimeter and the area of ΔABC.



4. Let R be the region consisting of the points (x, y) of the Cartesian plane satisfying both $|x| - |y| \le 1$ and $|y| \le 1$. Sketch the region R and find its area.

5. A rectangle, $HOMF$, has sides $HO = 11$ and $OM = 5$. A triangle ABC has H as the intersection of the altitudes, O the center of the circumscribed circle, M the midpoint of BC, and F the foot of the altitude from A. What is the length of BC?



6. Let C_1 and C_2 be circles whose centers are 10 units apart, and whose radii are 1 and 3. Find, with proof, the locus of all points M for which there exist points X on C_1 and Y on C_2 such that M is the midpoint of the line segment $\overline{XY}$.