Putnam Seminar 2008, Problem Set 5
Topic: Real Analysis, Part II

Sequences and Series

#1 Define the sequence $\{a_n\}_0^\infty$ by $a_0 = 0$, $a_1 = 1$, $a_2 = 2$, $a_3 = 6$, and

$$a_{n+4} = 2a_{n+3} + a_{n+2} - 2a_{n+1} - a_n, \quad \text{for } n \geq 0.$$

Prove that n divides a_n for all $n \geq 1$.

#2 Prove that for $n \geq 2$, the equation $x^n + x - 1$ has a unique root in the interval $[0,1]$. If x_n denotes this root, prove that the sequence $\{x_n\}_2^\infty$ is convergent and find its limit.

#3 Evaluate in closed form

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{m!n!}{(m+n+2)!}.$$

Continuity, Derivatives, and Integrals

#4 Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $|f(x) - f(y)| \geq |x - y|$ for all $x, y \in \mathbb{R}$. Prove that the range of f is all of $\mathbb{R}$.

#5 For a positive integer n, compute the integral

$$\int \frac{x^n}{1 + x + (x^2/2!) + \cdots + (x^n/n!)}\, dx.$$

Multivariable Calculus

Cauch-Schwartz Inequality:

$$\left(\int_D |f(x)g(x)|\ dx\right)^2 \leq \left(\int_D f(x)^2\, dx\right)\left(\int_D g(x)^2\, dx\right).$$

Minkowski's Inequality: If $p > 1$, then

$$\left(\int_D |f(x)+g(x)|\ dx\right)^{1/p} \leq \left(\int_D |fx|^p\ dx\right)^{1/p} + \left(\int_D |gx|^p\ dx\right)^{1/p}.$$

Hölder's Inequality: If $x_1, x_2, \ldots, x_n$, $y_1, \ldots, y_n$, p, and q are positive numbers with $1/p + 1/q = 1$, then

$$\sum_{i=1}^{n} x_i y_i \leq \left(\sum_{i=1}^{n} x_i^p\right)^{1/p}\left(\sum_{i=1}^{n} y_i^q\right)^{1/q} \text{ or } \int_D |f(x)g(x)|\ dx \leq \left(\int_D |fx|^p\ dx\right)^{1/p}\left(\int_D |gx|^q\ dx\right)^{1/q}.$$

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Multivariable Calculus

The Generalized Mean Inequality: Given the positive numbers $x_1, \dots, x_n$ and the positive weights $\lambda_1, \dots, \lambda_n$ with $\lambda_1 + \cdots + \lambda_n = 1$, the following inequality holds:

$$\lambda_1 x_1 + \cdots + \lambda_n x_n \geq x_1^{\lambda_1} \cdots x_n^{\lambda_n}.$$

The Lagrange Multipliers Theorem: If a function $f(x,y,z)$ subject to the constraint $g(x,y,z) = C$ has a maximum or a minimum, then this maximumum or minimum occurs at a point (a,b,c) of the set: $g(x,y,z) = C$ for which the gradients of f and g are parallel, i.e., for some real number λ,

$$f_x(a,b,c) = \lambda \cdot g_x(a,b,c), \dots, f_z(a,b,c) = \lambda \cdot g_z(a,b,c).$$

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#6 Find the global minimum of the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) = x^4 + 6x^2y^2 + y^4 - \frac{9}{4}x - \frac{7}{4}y.$$

#7 Prove that for $\alpha, \beta, \gamma \in [0, \pi/2)$,

$$\tan\alpha + \tan\beta + \tan\gamma \leq \frac{2}{\sqrt{3}} \sec\alpha \sec\beta \sec\gamma.$$

#8 Let $a_1 \leq a_2 \leq \cdots \leq a_n = m$ be positive integers. Denote by b_k the number of those a_i for which $a_i \geq k$. Prove that

$$a_1 + \cdots + a_n = b_1 + \cdots + b_m.$$

Functional Equations

#9 Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous nonzero function, satisfying the equation

$$f(x+y) = f(x)f(y) \quad \text{for all } x, y \in \mathbb{R}.$$

Prove that there exists $c > 0$ such that $f(x) = c^x$ for all $x \in \mathbb{R}$.

#10 Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ with the property that

$$f\big(f(x)\big) - 2f(x) + x = 0, \quad \text{for all } x \in \mathbb{R}.$$