Putnam Seminar 2008, Problem Set 4: Oct 8 – Oct 15
Topic: Real Analysis, Part I

Sequences and Series

Weierstrass' theorem: *A monotone bounded sequence of real numbers is convergent.*

Cauchy's criterion for convergence: *A sequence* $\{x_n\}_1^\infty$ *of points in* $\mathbb{R}$ *is convergent if and only if for any* $\varepsilon > 0$ *there is a positive integer* $N = N_\varepsilon$ *such that whenever* $n, m \geq N$, $|x_n - x_m| < \varepsilon$.

Let $\{a_n\}_1^\infty$ *and* $\{b_n\}_1^\infty$ *be two positive sequences.*

Comparison Tests: *If* $a_n/b_n \to c > 0$, *then the convergences of* $\sum_1^\infty a_n$ *and* $\sum_1^\infty b_n$ *are equivalent.*

Ratio Test: *Suppose that* $a_{n+1}/a_n \to A \neq 1$. *If* $A < 1$, *the series converges, and If* $A > 1$, *the series diverges.*

Root Test: *Suppose that* $\sqrt[n]{a_n} \to A \neq 1$. *If* $A < 1$, *the series converges, and If* $A > 1$, *the series diverges.*

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#1 Find a formula in compact form for the general term of the sequence defined recursively by $x_1 = 1$, $x_n = x_{n-1} + n$ if n is odd, and $x_n = x_{n-1} + n - 1$ if n is even.

#2 Compute

$$\lim_{n\to\infty} \left|\sin\left(\pi\sqrt{n^2+n+1}\right)\right|.$$

#3 Prove that the following sequence is convergent:

$$a_n = \sqrt{1+\sqrt{2+\sqrt{3+\cdots+\sqrt{n}}}}, \quad n \geq 1.$$

#4 Show that the following series converges when $|x| > 1$, and in this case find its sum:

$$\frac{1}{1+x} + \frac{2}{1+x^2} + \frac{4}{1+x^4} + \cdots + \frac{2^n}{1+x^{2^n}} + \cdots.$$

#5 The number q ranges over all possible powers with both the base and the exponent positive integers greater than 1, assuming each such value only once. Prove that

$$\sum_q \frac{1}{q-1} = 1.$$

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Continuity, Derivatives, and Integrals

Peano's Theorem: There exists a continous surjection $\phi : [0,1] \to [0,1] \times [0,1]$.

Intermediate Value Property: If k is a number in between $f(a)$ and $f(b)$ where f is continuous over $[a,b]$, then the graph of $y = f(x)$ over $[a,b]$ intersects $y = k$.

Theorem: There exists a function $f : [0,1] \to [0,1]$ that has the intermediate value property and is discontinuous at every point.

Mean Value Theorem: If f is differentiable, then given two points a and b, there is c in (a,b) such that $\big(f(b) - f(a)\big)/(b-a) = f'(c)$.

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#6 Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying $f(x) = f(x^2)$ for all $x \in \mathbb{R}$. Prove that f is constant.

#7 Prove that any convex polygonal surface can be divided by two perpendicular lines into four regions of equal area.

#8 Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = (x - a_1)(x - a_2) + (x - a_2)(x - a_3) + (x - a_3)(x - a_1)$$

with a_1, a_2, a_3 real numbers. Prove that $f(x) \geq 0$ for all real numbers x if and only if $a_1 = a_2 = a_3$.

#9 Let $P(x)$ be a polynomial with real coefficients such that for every positive integer n, the equation $P(x) = n$ has at least one rational root. Prove that $P(x) = ax + b$ with a and b rational numbers.

#10 Compute the integral

$$\int \frac{x^2+1}{x^4 - x^2 + 1}\, dx.$$