

Putnam Seminar 2008  
Problem Set 3: Sept. 10-17  
Algebra and Number Theory

1. Let  $S$  be a set of real numbers that is closed under multiplication (that is, if  $a$  and  $b$  are in  $S$ , then so is  $ab$ ). Let  $T$  and  $U$  be disjoint subsets of  $S$  whose union is  $S$ . Given that the product of any *three* (not necessarily distinct) elements of  $T$  is in  $T$  and that the product of any three elements of  $U$  is in  $U$ , show that at least one of the two subsets  $T$ ,  $U$  is closed under multiplication.
2. Let  $S$  be a nonempty set with an associative operation that is left and right cancellative ( $xy = xz$  implies  $y = z$ , and  $yx = zx$  implies  $y = z$ ). Assume that for every  $a$  in  $S$  the set  $\{a^n : n = 1, 2, 3, \dots\}$  is finite. Must  $S$  be a group?

3. Let  $G$  be a finite group of order  $n$  generated by  $a$  and  $b$ . Prove or disprove: there is a sequence

$$g_1, g_2, g_3, \dots, g_{2n}$$

such that

- (1) every element of  $G$  occurs exactly twice, and
  - (2)  $g_{i+1}$  equals  $g_i a$  or  $g_i b$ , for  $i = 1, 2, \dots, 2n$ . (Interpret  $g_{2n+1}$  as  $g_1$ .)
4. A *composite* (positive integer) is a product  $ab$  with  $a$  and  $b$  not necessarily distinct integers in  $\{2, 3, 4, \dots\}$ . Show that every composite is expressible as  $xy + xz + yz + 1$ , with  $x$ ,  $y$ , and  $z$  positive integers.
  5. How many primes among the positive integers, written as usual in base 10, are such that their digits are alternating 1's and 0's, beginning and ending with 1?
  6. Prove that there exist infinitely many integers  $n$  such that  $n$ ,  $n + 1$ , and  $n + 2$  are each the sum of two squares of integers. [Example:  $0 = 0^2 + 0^2$ ,  $1 = 0^2 + 1^2$ , and  $2 = 1^2 + 1^2$ .]