Putnam Seminar 2008, Problem Set 2: Sep 3 – 10
Topic: Mathematical Induction and The Pigeon Hole Principle

Mathematical Induction

Example 1: Use the Euclidean algorithm to prove that all positive integers have unique prime-power factorizations.

Example 2: Finitely many lines divide the plane into regions. Show that these regions can be colored by two colors in such a way that neighboring regions have different colors.

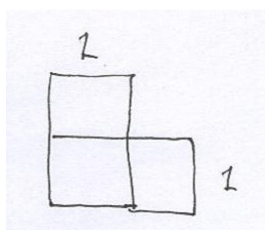
#1 Prove for all positive integers n the identity

$$\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}=1-\frac{1}{2}+\frac{1}{3}-\cdots+\frac{1}{2n-1}-\frac{1}{2n}.$$

#2 Let n be a positive integers. Prove that

$$1+\frac{1}{2^3}+\frac{1}{3^3}+\cdots+\frac{1}{n^3}<\frac{3}{2}.$$

#3 Prove that for any $n \geq 1$, a $2^n \times 2^n$ checkerboard with 1×1 corner square removed can be tiled by pieces of the form described in the figure below:



#4 Prove that for any positive integer $n \geq 2$ there is a positive integer m that can be written simultaneously as a sum of $2, 3, \ldots, n$ squares of nonzero integers.

#5 Show that every positive integer can be written as a sum of distinct terms of the Fibonacci sequence: $F_1 = 1,\ F_2 = 1,\ F_{n+2} = F_{n+1} + F_n$.

#6 Prove that any positive integer can be represented as $\pm 1^2 \pm 2^2 \pm \cdots \pm n^2$ for some positive integer n and some choice of the signs.

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The Pigeon Hole Principle

> *If $kn+1$ objects are distributed among n boxes, one of the boxes will contain at least $(k+1)$ objects.*

Example 1: Prove that every set of 10 two-digit integer numbers has two disjoint subsets with the same sum of elements.

Example 2: Given nine points inside the unit square, prove that some three of them form a triangle whose area does not exceed $1/8$.

#7 Given 50 distinct positive integers strictly less than 99, prove that some two of them sum to 99.

#8 In each of the unit squares of a 10×10 checkerboard, a positive integer not exceeding 10 is written. Any two numbers that appear in adjacent or diagonally adjacent squares of the board are relatively prime. Prove that some number appears at least 17 times.

#9 There are n people at party. Prove that there are two of them such that of the remaining $n-2$ people, there are at least $\lfloor n/2 \rfloor - 1$ of them each of whom knows both or else knows neither of the two.

#10 Show that any convex polyhedron has two faces with the same number of edges.