Putnam Seminar 2008, Problem Set 1: Aug 27 - Sep 3
Topic: Argument by Contradiction

Example 1: Prove Euclid's Theorem: *There are infinitely many prime numbers.*

Exercise 1: Prove that there are infinitely many prime numbers $p \equiv 3 \bmod 4$.

Example 2: Prove that there is no polynomial

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

with integer coefficients and of degree at least 1 with the property that $P(0)$, $P(1)$, $P(2)$, ... are all prime numbers.

Example 3: Let $F = \{E_1, E_2, \ldots, E_s\}$ be a family of subsets with r elements of some set X. Show that if the intersection of any $r+1$ (not necessarily distinct) sets in F is nonempty, then the intersection of all sets in F is nonempty.

#1 Prove that $\sqrt{2}+\sqrt{3}+\sqrt{5}$ is an irrational number.

#2 Show that no set of nine consecutive integers can be partitioned into two sets with the product of the elements of the first set equal to the product of the elements of the second set.

#3 Find the least positive integer n such that any set of n pairwise relatively prime integers greater than 1 and less than 2005 contains at least one prime number.

#4 Every point of three-dimensional space is colored red, green, or blue. Prove that one of the colors attains all distances, meaning that any positive real number represents the distance between two points of this color.

#5 Show that there does not exist a function $f : \mathbb{Z} \to \{1,2,3\}$ satisfying $f(x) \neq f(y)$ for all $x, y \in \mathbb{Z}$ such that $|x-y| \in \{2,3,5\}$.

#6 Show that there does not exist a strictly increasing function $f : \mathbb{N} \to \mathbb{N}$ satisfying $f(2) = 3$ and $f(mn) = f(m)f(n)$ for all $m, n \in \mathbb{N}$.

#7 Determine all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying

$$x\,f(y) + y\,f(x) = (x+y)\,f(x^2+y^2)$$

for all positive integers x and y.

#8 Show that the interval $[0,1]$ cannot be partitioned into two disjoint sets A and B such that $B = A + a$ for some real number a.

#9 Let $n > 1$ be an arbitrary real number and let k be the number of positive prime numbers less than or equal to n. Select $k+1$ positive integers such that none of them divides the product of all the others. Prove that there exists a number among the chosen $k+1$ that is bigger than n.