

Putnam Seminar 2008, Problem Set 4
Old Putnam Competition Problems

A1. Develop necessary and sufficient conditions which ensure that r_1, r_2, r_3 and r_1^2, r_2^2, r_3^2 are simultaneously roots of the equation $x^3 + ax^2 + bx + c = 0$.

A2. Let n be a given positive integer. How many solutions are there in ordered positive integer pairs (x, y) to the equation

$$\frac{xy}{x+y} = n?$$

A3. Assume that $f(x)$ is continuous in the interval $[0, 1]$, prove that

$$\int_{x=0}^{x=1} \int_{y=x}^{y=1} \int_{z=x}^{z=y} f(x) f(y) f(z) dz dy dx = \frac{1}{3!} \left(\int_{t=0}^{t=1} f(t) dt \right)^3.$$

A4. Let C be the unit circle $x^2 + y^2 = 1$. A point p is chosen randomly on the circumference of C and another point q is chosen randomly from the interior of C (these points are chosen independently and uniformly over their domains). Let R be the rectangle with sides parallel to the x - and y -axes with diagonal pq . What is the probability that no point of R lies outside of C ?

A5. Consider the arithmetic progression

$$a, a + d, a + 2d, \dots$$

where a and d are positive integers. For any positive integer k , prove that the progression has either no exact k th powers or infinitely many.

A6. For each positive integer n , let $a(n)$ be the number of zeros in the base 3 representation of n . For which positive real numbers x does the series

$$\sum_{n=1}^{\infty} \frac{x^{a(n)}}{n^3}$$

converge?

B1. Prove or disprove: if x and y are real numbers with $y \geq 0$ and $y(y+1) \leq (x+1)^2$, then $y(y-1) \leq x^2$.

B2. A not uncommon calculus mistake is to believe that the product rule for derivatives says that $(fg)' = f'g'$. If $f(x) = e^{x^2}$, determine, with proof, whether there exists an open interval (a, b) and a nonzero function g defined on (a, b) such that this wrong product rule is true for x in (a, b) .

B3. Let F be a field in which $1 + 1 \neq 0$. Show that the set of solutions to the equation $x^2 + y^2 = 1$ with x and y in F is given by $(x, y) = (1, 0)$ and

$$(x, y) = \left(\frac{r^2 - 1}{r^2 + 1}, \frac{2r}{r^2 + 1} \right),$$

where r runs through the elements of F such that $r^2 \neq -1$.

B4. Let k be the smallest positive integer with the following property:

There are distinct integers m_1, \dots, m_5 such that the polynomial

$$p(x) = (x - m_1) \cdots (x - m_5)$$

has exactly k nonzero coefficients.

Find, with proof, a set of integers m_1, \dots, m_5 for which this minimum k is achieved.

B5. For positive integers n , let $C(n)$ be the number of representations of n as a sum of nonincreasing powers of 2 where no power can be used more than three times. For example, $C(8) = 5$ since the representations for 8 are:

$$8, 4 + 4, 4 + 2 + 2, 4 + 2 + 1 + 1, \text{ and } 2 + 2 + 2 + 1 + 1.$$

Prove or disprove that there is a polynomial $P(x)$ such that $C(n) = [P(n)]$ for all positive integers n ; here $[u]$ denotes the greatest integers less than or equal to u .

B6. A “spherical ellipse” with foci A and B on a given sphere is defined as the set of all points P on the sphere such that $\widetilde{PA} + \widetilde{PB} = \text{constant}$. Here \widetilde{PA} denotes the shortest distance on the sphere between P and A . Determine the entire class of real spherical ellipses which are circles.