AUTOMORPHIC FORMS AND METAPLECTIC GROUPS

MARK BUDDEN [1]

3/25/05

1. Classical Automorphic Forms

Let $\mathcal{H}$ denote the Poincaré upper-half plane:

$$\mathcal{H} = \{x + iy \in \mathbb{C} \mid y > 0\}.$$

Then the group $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{C}$ by linear fractional transformation. Ie., if $g \in \mathrm{SL}_2(\mathbb{R})$ and $z \in \mathcal{H}$, then the action of g on z is given by

$$gz = \begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}.$$

An important subgroup of $\mathrm{SL}_2(\mathbb{R})$ on which we will focus our attention is the group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$.

If k is an even nonnegative integer, then a *modular form of weight k for* Γ is a holomorphic function $f : \mathcal{H} \longrightarrow \mathbb{C}$ that satisfies the equation

$$f(gz) = (cz + d)^k f(z)$$

for all $g \in \Gamma$ and $z \in \mathcal{H}$ and which is holomorphic at ∞ (see [Bm], §1.3 or [S1], Chap. VII, §2). Note that since $\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \in \Gamma$, we have $f(z + 1) = f(z)$ for all $z \in \mathcal{H}$. Therefore, any such function f has a Fourier expansion:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n z}.$$

If $a_n = 0$ for $n < 0$, then we say that f is *holomorphic at* ∞. If f is holomorphic at ∞ and $a_0 = 0$, then f is called a *cusp form*.

In 1952, Gelfand and Fomin [GF] noted a relationship between modular forms and representations of $\mathrm{SL}_2(\mathbb{R})$. This came about through the realization that the stabilizer of $i \in \mathcal{H}$ is the group

$$\mathrm{SO}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \middle| \ \mathrm{a}, \mathrm{b} \in \mathbb{R} \ \text{ and } \ \mathrm{a}^2 + \mathrm{b}^2 = 1 \right\}$$

leading to the identification of $\mathcal{H}$ with $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$. If f is a modular form of weight k for Γ, it defines a function ϕ_f on $\mathrm{SL}_2(\mathbb{R})$ by

$$\phi_f(g) := (cz + d)^{-k} f\left(\frac{ai + b}{ci + d}\right) \quad \text{where} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}).$$

One particular property (among many) of such functions is that

$$\phi_f(\gamma g) = \phi(g), \quad \forall \gamma \in \Gamma$$

so that ϕ_f can be thought of as living in $L^2(\mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z}))$. These functions arise in the representation spaces of $\mathrm{SL}_2(\mathbb{R})$ (see [Ge] for details).

---

[1] Partially supported by AASU Internal Grant #727188.

This realization led to a more general definition of an automorphic form over GL_2 and later over GL_r (see [Mu] for definitions). Metaplectic forms were also defined (see [KP]) as being automorphic forms defined on topological covering groups of GL_r. Such groups are called *metaplectic groups* and are the focus of this exposition.

2. Local Fields

Throughout, let $\mathbb{F}$ denote a nonarchimedean local field containing μ_n, the complete group of n^{th} roots of unity. Once and for all, we fix an identification $\varepsilon : \mu_n \hookrightarrow \mathbb{C}^\times$ into the group of n^{th} roots of unity in $\mathbb{C}$. With respect to a normalized valuation $\upsilon : \mathbb{F} \longrightarrow \mathbb{Z} \cup \{\infty\}$ (satisfying $\upsilon(xy) = \upsilon(x) + \upsilon(y)$ and $\upsilon(0) = \infty$), define the following sets

$$\begin{aligned} \mathcal{O} &= \{x \in \mathbb{F} \mid \upsilon(x) \geq 0\} \quad \text{(ring of integers in } \mathbb{F}),\\ \wp &= \{x \in \mathbb{F} \mid \upsilon(x) > 0\} \quad \text{(unique maximal ideal in } \mathcal{O}),\\ \mathcal{O}^\times &= \{x \in \mathbb{F} \mid \upsilon(x) = 0\} \quad \text{(units in } \mathcal{O}). \end{aligned}$$

If f is the residue degree defined using the degree of the residue field $\#(\mathcal{O}/\wp) = p^f = q$, then the absolute value on $\mathbb{F}$ is given by $|x|_\mathbb{F} = q^{-\upsilon(x)}$.

A well-known result of class field theory (see [W], Chap. VIII, §5) is the existence and uniqueness of the n^{th} *order local Hilbert symbol*

$$(\cdot,\cdot)_\mathbb{F} : \mathbb{F}^\times \times \mathbb{F}^\times \longrightarrow \mu_n$$

satisfying

$$\begin{aligned} &(aa', b)_\mathbb{F} = (a, b)_\mathbb{F}\ (a', b)_\mathbb{F}, \quad \forall a, a', b \in \mathbb{F}^\times,\\ &(a, b)_\mathbb{F}\ (b, a)_\mathbb{F} = 1, \quad \forall a, b \in \mathbb{F}^\times,\\ &(a, -a)_\mathbb{F} = 1 = (a, 1-a)_\mathbb{F}, \quad \forall a \in \mathbb{F}^\times, a \neq 1, \text{ and}\\ &\{x \in \mathbb{F}^\times \mid (x, y)_\mathbb{F} = 1,\ \forall y \in \mathbb{F}^\times\} = \mathbb{F}^{\times n}, \end{aligned}$$

where

$$\mathbb{F}^{\times n} := \{x \in \mathbb{F}^\times \mid x = y^n, \text{ for some } y \in \mathbb{F}^\times\}.$$

For our purposes, we assume that the Hilbert symbol is unramified (ie., $(x, y)_\mathbb{F} = 1$ for all $x, y \in \mathcal{O}^\times$). This is equivalent to assuming $|n|_\mathbb{F} = 1$.

Hilbert's Reciprocity Law (see [BH], Theorem 1.2) Let $\mathbb{K}$ be a global field, υ any place of $\mathbb{K}$, and $\mathbb{K}_\upsilon$ the corresponding completion. If $(\cdot,\cdot)_{\mathbb{K}_\upsilon}$ denotes the local Hilbert symbol and $a, b \in \mathbb{K}^\times$, then

$$\prod_\upsilon (a, b)_{\mathbb{K}_\upsilon} = 1.$$

3. Metaplectic Groups

Although one can define a metaplectic cover of $\mathrm{GL}_\mathrm{r}(\mathbb{F})$, for simplicity, we will focus on the case of $G = \mathrm{GL}_2(\mathbb{F})$. The *n-fold (0-twisted) metaplectic group* $\widetilde{G} = \widetilde{\mathrm{GL}}_2(\mathbb{F})$ is a nontrivial central extension of G by μ_n:

$$(1) \longrightarrow \mu_n \xrightarrow{\mathbf{i}} \widetilde{G} \xrightarrow{\mathbf{p}} G \longrightarrow (1). \tag{1}$$

As a set, the elements of $\widetilde{G}$ are of the form (g, ζ) where $g \in G$ and $\zeta \in \mu_n$. However, $\widetilde{G} \not\cong G \times \mu_n$ since multiplication is given by

$$(g, \zeta)(g', \zeta') = (gg', \zeta\zeta'\sigma(g, g'))$$

where $\sigma : G \times G \longrightarrow \mu_n$ is a nontrivial 2-cocycle (factor set) in $Z^2(G; \mu_n)$. This 2-cocycle was first described in a method suitable for computation by Kubota [K], who showed that

$$\sigma(g, g') = \left(\frac{X(gg')}{X(g)}, \frac{X(gg')}{X(g')\det(g)} \right)_{\mathbb{F}} \tag{2}$$

where

$$X\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{cases} c & \text{if} \quad c \neq 0 \\ d & \text{if} \quad c = 0. \end{cases}$$

For a description of the cocycle σ when $G = \mathrm{GL}_\mathrm{r}(\mathbb{F})$, see [M] and [BLS]. One can also define a *c-twisted metaplectic cover* using the cocycle

$$\sigma_c(g, g') = \sigma(g, g')(\det(g), \det(g'))_{\mathbb{F}}^{c},$$

where $c \in \mathbb{Z}/n\mathbb{Z}$, but we will focus on 0-twisted covers in this exposition. In the short exact sequence (1), the inclusion map $\mathbf{i} : \mu_n \longrightarrow \widetilde{G}$ is given by $\zeta \mapsto (1, \zeta)$ and the projection map $\mathbf{p} : \widetilde{G} \longrightarrow G$ is given by $(g, \zeta) \mapsto g$. We also define the section $\mathbf{s} : G \mapsto \widetilde{G}$ by $g \mapsto (g, 1)$.

4. Subgroups

Consider the following subgroups of G:

$$B = \left\{ \begin{pmatrix} a & b \\ & d \end{pmatrix} \;\middle|\; a, d \in \mathbb{F}^\times,\ b \in \mathbb{F} \right\} \qquad \text{(Borel subgroup of } G\text{)}$$

$$T = \left\{ \begin{pmatrix} a & \\ & d \end{pmatrix} \;\middle|\; a, d \in \mathbb{F}^\times \right\} \qquad \text{(Torus in } B\text{)}$$

$$N = \left\{ \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \;\middle|\; b \in \mathbb{F} \right\} \qquad \text{(Unipotent radical of } B\text{)}$$

$$W = \left\{ \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \right\} \qquad \text{(Weyl subgroup of } G\text{)}$$

There is a unique decomposition $B = TN$ and a further decomposition (Iwasawa decomposition)

$$G = BK = TNK,$$

where $K = \mathrm{GL}_2(\mathcal{O})$ is the maximal compact subgroup of G. Another decomposition worth noting is the Bruhat decomposition:

$$G = B \cup Bw_0B,$$

where $w_0 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$ is the 'long element' of W and the union is disjoint.

Now we consider the corresponding subgroups of $\widetilde{G}$. If H is a subgroup of G, then let $\widetilde{H} = \mathbf{p}^{-1}(H)$, the metaplectic preimage of H under $\mathbf{p}$. We form the groups $\widetilde{T}$ and $\widetilde{B}$ in this way. To define the subgroups of $\widetilde{G}$ analogous to N and W, we need the following definition. For any subgroup H of G, $\widetilde{G}$ *splits* over H if there exists a homomorphism $\mathbf{h} : H \longrightarrow \widetilde{G}$ such that the composition

$$(\mathbf{p} \circ \mathbf{h}) : H \longrightarrow H$$

is the identity. In this case, there is a "copy" of H in $\widetilde{G}$. The cocycle (2) is trivial on $N \times N$ and hence, $\widetilde{G}$ splits over N via the preferred section $\mathbf{s}$. Thus, we can identify $\mathbf{s}(N)$ with N, and we denote $N^s := \mathbf{s}(N)$. Similarly, $\widetilde{G}$ splits via $\mathbf{s}$ over W and we define $W^s := \mathbf{s}(W)$.

The group $\widetilde{G}$ is an example of an *ℓ-group*: a topological group with a fundamental system of neighborhoods of the identity consisting of open, compact subgroups. By a *topological group*, we mean a group that is a topological space along with a continuous map

$$(x, y) \mapsto x^{-1}y.$$

The group $\widetilde{G}$ splits over K via a canonical splitting $\mathbf{k} : K \longrightarrow \widetilde{G}$ (shown in [Mo]). We will not describe $\mathbf{k}$ here, but it will be enough to note that

$$\mathbf{k}|_{T\cap K} = \mathbf{s}|_{T\cap K}, \quad \mathbf{k}|_W = \mathbf{s}|_W, \quad \text{and} \quad \mathbf{k}|_{N\cap K} = \mathbf{s}|_{N\cap K}.$$

Let $K^k := \mathbf{k}(K)$ and for every $m > 0$ define $K_m^k := \mathbf{k}(K_m)$ where

$$K_m := \{x \in K \mid x \equiv I (\text{mod } \wp^m)\}.$$

The topology of $\widetilde{G}$ is given by taking the collection K_m^k as a basis of open, compact neighborhoods of the identity in $\widetilde{G}$.

5. REPRESENTATION THEORY

A *complex representation* (π, V) of an arbitrary ℓ-group $\overline{G}$ is a homomorphism

$$\pi : \overline{G} \longrightarrow \text{Aut}(V)$$

where $\text{Aut}(V)$ is the group of automorphisms of a complex vector space V. To simplify the notation, (π, V) is sometimes denoted by just π. The *dimension* of a representation π is the dimension of V over $\mathbb{C}$.

To avoid working directly with the topology of a representation space V of an ℓ-group $\overline{G}$, we introduce the concept of an admissible representation. A representation (π, V) is called *smooth* (or *algebraic*) if the stabilizer

$$\text{Stab}(v) = \{g \in \overline{G} \mid \pi(g)v = v\}$$

is open in $\overline{G}$ for every $v \in V$. It is called *admissible* if it is smooth and for every open subgroup H in $\overline{G}$, the set of H-fixed vectors

$$V^H = \{v \in V \mid \pi(h)v = v, \ \forall h \in H\}$$

is finite dimensional. When working with admissible representations of an ℓ-group, one is able to avoid dealing with the topology of V and can focus on the representation theory using only algebraic methods.

Given an ℓ-group $\overline{G}$ and a representation (π, V) of $\overline{G}$, one may ask if π can be reduced to a vector space W of dimension less than that of V. A subspace $W \subseteq V$ is called *$\overline{G}$-stable* if

$$\pi(g)W \subseteq W$$

for all $g \in \overline{G}$. A vector space V is called *irreducible* if it does not contain any $\overline{G}$-stable subspaces other than $\{0\}$ and V itself. Otherwise, it is called *reducible.*

6. PRINCIPAL SERIES REPRESENTATIONS

A principal series representation is a representation that is induced from an irreducible representation of a minimal parabolic subgroup (ie., a Borel subgroup). To define such a representation we will need to make use of the following theorem (see [Bm] or [BG]).

Stone-von Neumann Theorem Let δ be a genuine quasicharacter of the center of $\widetilde{T}$. If δ is extended to any maximal abelian subgroup and then induced to $\widetilde{T}$, the resulting representation is irreducible. The isomorphism class of this representation depends only on δ, not on the choice of maximal abelian subgroup nor on its extension to this group.

We begin with the modular quasicharacter on B

$$\delta : B \longrightarrow \mathbb{C}^\times$$

given by

$$\delta\left(\begin{pmatrix} a & b \\ & d \end{pmatrix}\right) = \left|\frac{a}{d}\right|_{\mathbb{F}}$$

and use it to define a genuine character on the center of $\widetilde{T}$.

Theorem The center of $\widetilde{T}$ is the group

$$\widetilde{T}^n = \mathbf{p}^{-1}(T^n) := \mathbf{p}^{-1}\left\{\begin{pmatrix} x & \\ & y \end{pmatrix} \;\middle|\; x, y \in \mathbb{F}^{\times n}\right\}.$$

Thus, for $s \in \mathbb{C}^\times$, define the quasicharacter $\delta^s : \widetilde{T}^n \longrightarrow \mathbb{C}^\times$ by

$$\delta^s\left(\begin{pmatrix} x & \\ & y \end{pmatrix}, \zeta\right) = \zeta \left|\frac{x}{y}\right|_{\mathbb{F}}^s.$$

The maximal abelian subgroup that we extend δ^s to is given by $\widetilde{T}_* := \mathbf{p}^{-1}(T_*)$ where

$$T_* := \left\{\begin{pmatrix} x & \\ & y \end{pmatrix} \;\middle|\; x, y \in \mathbb{F}^\times \text{ and } \upsilon(x) \equiv \upsilon(y) \equiv 0 (\text{mod } n)\right\}.$$

Next, we extend δ^s to $\widetilde{B}_* = \widetilde{T}_* N^s$ by making it trivial on N^s. Finally, we define the representation (π_s, V_s) to be the induced representation

$$\mathrm{Ind}_{\widetilde{B}_*}^{\widetilde{G}}(\delta^{s+1/2}),$$

where

$$V_s = \{f \in C^\infty(\widetilde{G}) \mid f(bg) = \delta^{s+1/2}(b) f(g),\ \forall b \in \widetilde{B}_*,\ g \in \widetilde{G}\}$$

and π_s acts by right translation:

$$(\pi_s(g)) f(g') = f(g'g).$$

Although it is not applied directly, the Stone-von Neumann Theorem guarantees the uniqueness (up to isomorphism) of the representation (π_s, V_s).

Throughout these notes, the cover n has been fixed. Now we allow it to vary and denote the corresponding representation, as described in the previous section, by $(\pi_s, V_s)^{(n)}$. The following theorem is Theorem 4.2 of [Bu2] and relates the irreducibility of $(\pi_s, V_s)^{(n)}$ over different covers.

Theorem If d is any divisor of n and $|n|_{\mathbb{F}} = 1$, then

$$(\pi_s, V_s)^{(n)} \text{ irreducible} \quad \Longleftrightarrow \quad (\pi_{sn/d}, V_{sn/d})^{(d)} \text{ irreducible.} \tag{3}$$

7. Future Directions

The current focus of the author is to extend (3) to $\widetilde{\mathrm{GL}}_r(\mathbb{F})$ when $r > 2$. It is believed that this result should hold in exactly the same form, but the proof is much more complicated. For example, the 2-cocycle (2) described by Kubota [K] is only valid for $r = 2$. In [BLS], Banks, Levi, and Sepanski described a block-compatible 2-cocycle for $\widetilde{\mathrm{GL}}_r(\mathbb{F})$ which can be used, but is much more tedious to work with.

Ultimately, this work should be compiled to obtain global consequences. One can put together information from all of the local fields by considering the *ring of adèles* of a global field $\mathbb{F}$:

$$\mathbb{A}_{\mathbb{F}} := \prod_v (\mathbb{F}_v : \mathcal{O}_v).$$

In this definition, the direct product is restricted in the sense that all but finitely many of the entries (excluding the archimedean completions) are in $\mathcal{O}_v$. The representations π_v on $\widetilde{\mathrm{GL}}_r(\mathbb{F}_v)$ can be combined to form a representation

$$\pi = \bigotimes_v \pi_v$$

on $\widetilde{\mathrm{GL}}_r(\mathbb{A}_{\mathbb{F}})$. One motivation for making this extension is that the main result (3) was obtained by considering the zeros of local L-factors. When combined into the global case, we are no longer looking at L-factors, but rather the Euler Product of an L-function.

References

[B] W. Banks, *Exceptional Representations on the Metaplectic Group*, Dissertation, Stanford University (1994).

[BBL] W. Banks, D. Bump, and D. Lieman *Whittaker-Fourier Coefficients of Metaplectic Eisenstein Series*, Compositio Mathematica 135 (2003), pp. 153-178.

[BLS] W. Banks, J. Levy, and M. Sepanski, *Block-Compatible Metaplectic Cocycles*, J. Reine Angew. Math. 507 (1999), pp. 131-163.

[Bu1] M. Budden *On the Local Coefficients of Principal Series Representations of Metaplectic Groups*, Dissertation, University of Missouri-Columbia (2003).

[Bu2] M. Budden *Local Coefficient Matrices of Metaplectic Groups*, submitted (2004).

[Bm] D. Bump, "Automorphic Forms and Representations," Cambridge University Press, 1997.

[BG] D. Bump and D. Ginzburg, *Symmetric Square L-Functions on* GL(r), Annals of Mathematics 136 (1992), pp. 137-205.

[BH] D. Bump and J. Hoffstein, *On Shimura's Correspondence*, Duke Mathematical Journal Vol. 55, No. 3 (1987), pp. 661-691.

[Ge] S. Gelbart, *An Elementary Introduction to the Langlands Program*, Bull. of the Amer. Math Soc. Vol. 10, No. 2 (1984), pp. 177-219.

[GF] I. Gelfand and S. Fomin, *Geodesic Glows on Manifolds of Constant Negative Curvature*, Upsekhi Mat. Nauk. Vol. 7, No. 1 (1952), pp. 118-137.

[I] H. Iwaniec, "Topics in Classical Automorphic Forms," Graduate Studies in Mathematics 17, American Mathematical Society, 1997.

[KP] D. Kazhdan and S. J. Patterson, *Metaplectic Forms*, Publ. Math. IHES 59 (1984), pp. 35-142.

[K] T. Kubota, "Automorphic Forms and Reciprocity in a Number Field," Kiyokuniya Book Store, 1973.

[M] H. Matsumoto, *Sur les Sous-Groupes Arithmétiques des Groupes Simi-Simples Déployés*, Ann. Scient. Éc. Norm. Sup. 4^e Série (1969), pp. 1-62.

[Mo] C. Moore, *Group Extensions of p-adic and Adelic Linear Goups*, Publ. Math. IHES 35 (1968), pp. 157-222.

[Mu] R. Murty, *A Motivated Introduction to the Langlands Program*, from "Advances in Number Theory," Clarendon Press (1993), pp.37-66.

[N] J. Neukirch, "Algebraic Number Theory," A Series of Comprehensive Studies in Mathematics Vol. 322 (translated by Norbert Schappacher), Springer-Verlag, 1999.

[S1] J.-P. Serre, "A Course in Arithmetic," Graduate Texts in Mathematics 7, Springer-Verlag, 1973.

[S2] J.-P. Serre, "Local Fields," Graduate Texts in Mathematics 67, Springer-Verlag, 1979.

[W] A. Weil, "Basic Number Theory, 2nd ed.," Springer-Verlag, 1973.

Algebraic Number Theory: [N], [S1], [S2], [W]

Automorphic Forms: [Bm], [Ge], [GF], [I], [K], [Mu], [S1]

Metaplectic Groups: [B], [BBL], [BLS], [Bu1], [Bu2], [BG], [BH], [KP], [K], [M], [Mo]