# MATHEMATICS RESEARCH SEMINAR

3:30-4:30 October 7, 2005

104 University Hall

## RATIONAL RECIPROCITY LAWS

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*ABSTRACT:* Rational reciprocity laws refer to reciprocity laws which provide information about whether or not a rational prime is an $n^{th}$ power residue of another rational prime . In the case $n = 2$, we have the well-known Law of Quadratic Reciprocity, which is stated below. In 1934, Scholz proved a rational reciprocity law for the $n = 4$ case (also stated below) via class field theory. In this talk, we will prove Scholz's Reciprocity Law by investigating the splitting of the cyclotomic polynomial $\Phi_p(x)$ over the unique quadratic and quartic subfields of $\mathbb{Q}(\zeta_p)$. Additionally, we will discuss ways in which these techniques can be generalized to provide similar rational reciprocity laws when $n = 2^t$ for $t > 2$.

**Law of Quadratic Reciprocity** If $p$ and $q$ are distinct primes then

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{(p-1)(q-1)/4},$$

where

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } x^2 \equiv a (\text{mod } p) \text{ has a solution} \\ -1 & \text{otherwise.} \end{cases}$$

**Scholz's Reciprocity Law** If $p \equiv q \equiv 1(\text{mod } 4)$ are distinct primes such that $\left(\frac{p}{q}\right) = \left(\frac{q}{p}\right) = 1$, then

$$\left(\frac{p}{q}\right)_4 \left(\frac{q}{p}\right)_4 = \left(\frac{\varepsilon_p}{q}\right) = \left(\frac{\varepsilon_q}{p}\right),$$

where $\varepsilon_d$ is the fundamental unit (assuming $d > 0$) of the quadratic field $\mathbb{Q}(\sqrt{d})$ and the rational $4^{th}$ power residue symbol is defined by

$$\left(\frac{a}{p}\right)_4 = \begin{cases} 1 & \text{if } x^4 \equiv a (\text{mod } p) \text{ has a solution} \\ -1 & \text{otherwise.} \end{cases}$$