RATIONAL RECIPROCITY LAWS

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ABSTRACT: Rational reciprocity laws refer to reciprocity laws which provide information about whether or not a rational prime is an \( n^{th} \) power residue of another rational prime. In the case \( n = 2 \), we have the well-known Law of Quadratic Reciprocity, which is stated below. In 1934, Scholz proved a rational reciprocity law for the \( n = 4 \) case (also stated below) via class field theory. In this talk, we will prove Scholz’s Reciprocity Law by investigating the splitting of the cyclotomic polynomial \( \Phi_p(x) \) over the unique quadratic and quartic subfields of \( \mathbb{Q}(\zeta_p) \). Additionally, we will discuss ways in which these techniques can be generalized to provide similar rational reciprocity laws when \( n = 2^t \) for \( t > 2 \).

**Law of Quadratic Reciprocity** If \( p \) and \( q \) are distinct primes then

\[
\left( \frac{p}{q} \right) \left( \frac{q}{p} \right) = (-1)^{(p-1)(q-1)/4},
\]

where

\[
\left( \frac{a}{p} \right) = \begin{cases} 
 1 & \text{if } x^2 \equiv a \pmod{p} \text{ has a solution} \\
 -1 & \text{otherwise}
\end{cases}
\]

**Scholz’s Reciprocity Law** If \( p \equiv q \equiv 1 \pmod{4} \) are distinct primes such that \( \left( \frac{q}{p} \right) = \left( \frac{p}{q} \right) = 1 \), then

\[
\left( \frac{p}{q} \right)_4 \left( \frac{q}{p} \right)_4 = \left( \frac{\varepsilon_d}{q} \right)_4 = \left( \frac{\varepsilon_d}{p} \right)_4,
\]

where \( \varepsilon_d \) is the fundamental unit (assuming \( d > 0 \)) of the quadratic field \( \mathbb{Q}(\sqrt{d}) \) and the rational \( 4^{th} \) power residue symbol is defined by

\[
\left( \frac{a}{p} \right)_4 = \begin{cases} 
 1 & \text{if } x^4 \equiv a \pmod{p} \text{ has a solution} \\
 -1 & \text{otherwise}
\end{cases}
\]