On the enumeration of $k$-omino towers

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Abstract

We describe a class of geometric objects called $k$-omino towers that are created by stacking blocks of size $k \times 1 \times 1$ on a convex base composed of these same $k$-omino blocks. These towers are equivalent to a class of fixed polyominoes or lattice animals, through the creation of a polyomino using the boundary of the two-dimensional vertical face of the tower. By applying a partition to the set of $k$-omino towers of fixed area $kn$, we give a recurrence using towers consisting of one less $k$-omino and bases of varying lengths, therefore showing the set of $k$-omino towers is enumerated by a Gauss hypergeometric function. The proof in this case implies a more general hypergeometric identity with parameters similar to those given in a classical result of Kummer. Keywords. polyomino, Gauss hypergeometric function, towers

1. Introduction

If you ever played with LEGO®s or DUPLO®s as a child, you probably tried to stack the blocks into large towers, sometimes in a very haphazard way. An interesting enumerative question immediately arises: How many different towers can be created? In this manuscript, we consider blocks of fixed size; with unit width and height and with length $k$ units, called $k$-omino blocks. We ask, for a given $k$, how many towers may be created using $k$-omino blocks? It turns out, that these questions are related to a larger enumerative question with a rich history.

A polyomino is a collection of unit squares with incident sides. Solomon Golomb is credited with bringing the attention of the mathematical community to polyominoes through an article published in the American Mathematical Monthly in 1954 [8] and his book on polyominoes [9] published first in 1965. Researchers from various disciplines have been interested in polyominoes, from chemists and physicists, to statisticians and recreational mathematicians. Many polyomino problems lie in the area of mathematical problems that are easy to describe, but often surprisingly difficult to solve. For example, no formula is known for the number of polyominoes parametrized by area, although asymptotic bounds do exist, see [13], for example. This general case may be unknown, but much work has been done to enumerate certain classes of polyominoes using parameters such
as area, perimeter, length of base or top, number of rows or columns, and others. One common technique employed in these types of problems is to associate classes of polyominoes with words in algebraic languages, such as the classical association between parallelogram polyominoes of perimeter \((2n+2)\) and Dyck words of length \(2n\), the more recent associations by Barcucci, Frosini and Rinaldi in [1] who use ECO method to associate polyominoes bounded by rectangles with Grand-Dyck and Grand-Motzkin paths, or the work by Delest and Viennot [5] and Domoços [6] with these and other types of words. Another widely used technique is to dissect the polyomino into smaller parts. This technique is employed by Wright [16], Klarner and Rivest [13], and Bender [3]. In this paper, we utilize a third technique by employing a recursion on the base of a polyomino tower similar to that of Barcucci et. al [2] whose recursion was dependent on the rightmost column of a steep staircase polyomino.

In particular, we will consider fixed polyominoes, also called a fixed animals, which are oriented polyominoes such that different orientations of the same free shape are considered distinct. As the \(k\)-ominos all have fixed depth, they are equivalent to two-dimensional polyominoes. In particular, we study subsets of fixed polyominoes that have been derived from stacked towers of dominos or \(k\)-ominos. Section 2 defines and enumerates the class of polyominoes inspired by domino building blocks, which we call domino towers, and Section 3 generalizes these results to other horizontal \(k\)-omino blocks in terms of hypergeometric functions with the main result as follows:

1.1 Theorem. The number of \(k\)-omino towers with area \(nk\), \(D_k(n)\), is given by

\[
D_k(n) = \binom{kn - 1}{n - 1} \, _2F_1(1, 1 - n; (k - 1)n + 1; -1)
\]

where \(_2F_1(a, b; c; z)\) is the Gauss hypergeometric function and \(k, n \geq 1\).

2. Domino towers

In order to provide insight on the general case, we begin with the case where \(k = 2\). A domino is a 2-omino block which is two units in length and has two ends, a left end and a right end. Recall, a domino can refer to both the three-dimensional domino block and the two-dimensional domino polyomino through the correspondence of the block to the polyomino described by the boundary of the length two vertical face of the block. Similarly, the boundary of the vertical face of a collection of incident domino blocks may also define a fixed polyomino. In such a collection, a domino is in the base if no dominos are or could be underneath it, and the level of a domino will be the vertical distance from the base. Domino towers in terms of their area \(2n\) and base of length \(2b\) are defined as follows:

2.1 Definition. For \(n \geq b \geq 1\), an \((n, b)\)-domino tower is a fixed polyomino created by sequentially placing \(n - b\) dominos horizontally on a convex, horizontal base of composed of \(b\) dominos, such
that if a non-base domino is placed in position indexed by \( \{(x, y), (x + 1, y)\} \), then there must be a domino in position \( \{(x - 1, y - 1), (x, y - 1)\}, \{(x, y - 1), (x + 1, y - 1)\}, \text{or} \{(x + 1, y - 1), (x + 2, y - 1)\} \).

We note, that all dominos are placed with the same horizontal orientation in space so that the dimensions are 2 along the horizontal axis and 1 along the other axes. (See Durhuss and Eilers [7] for an exploration of \(2 \times 4\) LEGO® blocks where different orientations are allowed.) Finally, we define a \textit{column} of polyomino or the corresponding domino tower as the intersection of the polyomino with an infinite vertical line of unit squares. Now, using parameters area and base, domino towers may be counted with binomial coefficients.

\[2.2 \text{ Theorem.} \quad \text{The number of} \ (n, b) \text{-domino towers, } d_b(n), \text{ is given by } \binom{2n - 1}{n - b} \text{ for } n \geq b \geq 1.\]

\textit{Proof.} We proceed by induction on \( n \) and \( b \). Clearly, the number of \((1, 1)\)-domino towers is given by \( \binom{1}{0} = 1 \), so assume \( n > 1 \). Applying the recursive formula on binomial coefficients, for \( n \geq 2 \) we have

\[
\binom{2n - 1}{n - b} = \binom{2(n - 1) - 1}{(n - 1) - (b - 1)} + 2 \binom{2(n - 1) - 1}{(n - 1) - b} + \binom{2(n - 1) - 1}{(n - 1) - (b + 1)}. \tag{2.1}
\]

First, in the case where the base of a \(n\)-domino tower is a single domino, that is \( b = 1 \), the tower could have been created by placing a \((n - 1, 1)\)-domino tower on one of three positions on the base domino (left, right, or middle), or by centering a \((n - 1, 2)\)-domino tower on the single domino base. Hence the number of \((n, 1)\)-domino towers is given by three times the number \((n - 1, 1)\)-domino towers plus the number of \((n - 1, 2)\)-domino towers and

\[
3 \binom{2(n - 1) - 1}{n - 2} + \binom{2(n - 1) - 1}{n - 3} = \binom{2(n - 1) - 1}{n - 1} + 2 \binom{2(n - 1) - 1}{n - 2} + \binom{2(n - 1) - 1}{n - 3}
\]

satisfies the recurrence.

Now, assuming \( n \geq b > 1 \), we will show the \((n, b)\)-domino towers may be built from domino towers of \((n - 1)\) blocks and bases of length \(b - 1\), \(b\), and \(b + 1\). To begin, we partition the set of \((n, b)\)-domino towers into four disjoint sets as follows:

1. Let \( A_{n,b} \) be the set of \((n, b)\)-domino towers such that the leftmost domino on the first level does not intersect the column containing the left end of the leftmost domino of the base.

2. Let \( B_{n,b} \) be the set of \((n, b)\)-domino towers that have a domino on the first level whose left end intersects the column containing the left end of the leftmost domino in the base.

3. Let \( C_{n,b} \) be the set of \((n, b)\)-domino towers that have a domino placed on the first level so that its right end intersects the column containing the leftmost domino of the base and whose left end extends past the base on the left side. Further, assume the rightmost domino on the first level of the tower does not extend past the base on the right side, that is, the column containing the right end of the rightmost domino on the first level must intersect the base.
4. Let $D_{n,b}$ be the set of $(n,b)$-domino towers that have a domino placed on the first level so that its right end intersects the column containing the leftmost domino of the base and whose left end extends past the base on the left side. Further, the rightmost domino on the first level must extend one unit past the base on the right side, that is, the column containing the right end of the rightmost domino on the first level does not intersect the base.

These sets are illustrated in Figure 1 in the case of $n = 4$ and $b = 2$.

To begin, we observe in towers from the set $A_{n,b}$, all dominos on levels greater than zero can be supported by the rightmost $b - 1$ dominos in the base. Thus, the leftmost domino from the base can be removed resulting in a $(n - 1, b - 1)$-domino tower. Since the process can be reversed, the set $A_{n,b}$ is in bijection with the set of $(n - 1, b - 1)$-domino towers, and thus has cardinality $\binom{2n-3}{n-b}$.

Next, given a tower in the set $B_{n,b}$, we proceed by removing the leftmost domino from the first level. Any dominos being completely supported by this domino will shift down one level, leaving a $(n - 1, b)$-domino tower. Similarly, a new domino could be added to an $(n - 1, b)$-domino tower directly under the leftmost domino of the base, pushing up any dominos intersecting those two columns and creating a $(n, b)$-domino tower. Thus we have that $B_{n,b}$ is enumerated by $\binom{2n-3}{n-b-1}$.

Now, we wish to show that $C_{n,b}$ is in bijection with $(n - 1, b)$-domino towers by constructing a bijection with $B_{4,2}$. To construct the bijection, fix all dominos not contained in the base of the tower. Then shift the base one unit, or half of a domino, to the right. (For example, shifting the base in the row $B_{4,2}$ row in Figure 1 produces the row $C_{4,2}$ below it.) In the inverse map, the base
of a domino towers from the set \( C_{n,b} \) are shifted to the left by one unit. The map is well-defined because the second condition on \( C_{n,b} \) allows all dominos to still remain supported by the base. Hence, \( C_{n,b} \) has cardinality \( \binom{2n-1}{n-b} \).

Finally, consider the set \( D_{n,b} \). All towers in \( D_{n,b} \) have dominos on the first level extending over the base on both sides. To subtract the equivalent of a domino from a tower in \( D_{n,b} \), we remove the right end of the leftmost domino on the first level and the left end of the rightmost domino on the first level. The remaining ends of these dominos will drop down one level to bracket the base and all dominos completely supported by the removed dominos will also drop down. To complete the process replace the base of \( b \) dominos bracketed by unit squares with a base of \( b+1 \)-dominos; see Figure 2. As this process can also be reversed, we have a bijection between the set of \( (n-1,b+1) \)-domino towers and \( D_{n,b} \). Consequently, the cardinality of \( D_{n,b} \) is \( \binom{2n-3}{n-b-2} \). Thus, the claim has been proven.

As a consequence of Theorem 2.2, we have the following corollary.

2.3 Corollary. The number of \( n \)-domino towers is \( 4^n-1 \) for \( n \geq 1 \).

This formula may be found using the combinatorial identity 1.83 found in Gould [10].

We can also apply the recursion to determine the bivariate generating function for the number of \((n,b)\)-domino towers.

2.4 Proposition. The bivariate generating function \( D_2(x,y) \) for the number of \((n,b)\)-domino towers is

\[
D_2(x,y) = \sum_{n \geq 1} \sum_{b=1}^n d_b(n)x^n y^b = \sum_{n \geq 1} \sum_{b=1}^n \binom{2n-1}{n-b} x^n y^b = \frac{xy + (y-1)\frac{x}{y} \left( \frac{1-\sqrt{1-4x}}{2\sqrt{1-4x}} \right)}{1 - 2x - xy - \frac{x}{y}}.
\]

Proof. We apply the recurrence used in the proof of Theorem 2.2.

\[
D_2(x,y) = \sum_{n \geq 1} \sum_{b=1}^n d_b(n)x^n y^b = \sum_{n \geq 1} \left( \frac{2(n-1)-1}{(n-1)-(b-1)} \right) x^n y^b + 2 \left( \frac{2(n-1)-1}{(n-1)-(b+1)} \right) x^n y^b = xy + xyD_2(x,y) + y \sum_{n \geq 1} \left( \frac{2(n-1)-1}{n-1} \right) x^n + 2xD_2(x,y) + \frac{x}{y}D_2(x,y) - \sum_{n \geq 1} \left( \frac{2(n-1)-1}{n-2} \right) x^n
\]
Because
\[
\sum_{n \geq 1} \left( \frac{2(n-1)}{n-1} \right) x^n = \sum_{n \geq 1} \left( \frac{2(n-1)}{n-2} \right) x^n = \frac{x}{2} \left( \frac{1 - \sqrt{1 - 4x}}{\sqrt{1 - 4x}} \right),
\]
the result follows.

2.5 Remark. There is another connection between domino towers and polyominoes through a natural bijection given by Viennot in [15] associating strict heaps of dominos and directed polyominoes. Bousquet-Mélou and Rechnitzer [4] enumerate these heaps to find large classes of polyominoes whose growth constants are larger than known classes. As domino towers are a subset of domino heaps one could use this bijection to describe a class of directed polyominoes whose growth constant in terms of area is \((4n - 1) \frac{1}{2^n} = (1/4) \frac{1}{2} \cdot 2 \sim 2\), this is, approximately half of that of the general fixed polyominoes which is estimated around 4.06 [11]. (This result is obtained through a numerical analysis of the series of fixed polyominoes of limited size.) However, we will not discuss this any further here as we are interested in the towers themselves as polyominoes.

3. k-omino towers

We generalize the results of Section 2 to horizontal polyominoes of integer length. As a reminder, a \(k\)-omino refers to both the \(k \times 1\) horizontal polyomino and the \(k \times 1 \times 1\) block, and not to a more general fixed polyomino with area \(k\).

3.1 Definition. For \(n \geq b \geq 1\), an \((n,b)\)-\(k\)-omino tower is a fixed polyomino created by sequentially placing \((n-b)\) horizontal \(k\)-omino on a convex, horizontal base composed of \(b\) \(k\)-omino, such that if a non-base \(k\)-omino is placed in position \(\{(x, y), (x+k-1, y)\}\), then there must be a \(k\)-omino in one of the \(2k-1\) positions between \(\{(x-k+1, y-1), (x, y-1)\}\) and \(\{(x+k-1, y-1), (x+2k-2, y-1)\}\).

Again, we do not allow for different orientations of the \(k\)-omino blocks in three-dimensional space. The following result generalizes Theorem 2.2.

3.2 Theorem. The number of \((n,b)\)-\(k\)-omino towers is given by \(\binom{kn-1}{n-b}\) for \(n, b \geq 1\) and \(k \geq 1\).

Proof. Repeated application of the recursion on binomial coefficients gives the equation
\[
\binom{kn-1}{n-b} = \sum_{i=0}^{k} \binom{k}{i} \binom{k(n-1) - 1}{(n-1) - (b+i-1)}.
\]

First, assume \(b \geq 2\). We will show \(k\)-omino towers can be built from towers of one less \(k\)-omino and bases of sizes from \(b-1\) to \(b+k-1\).

To define the map, begin with a \((n,b)\)-\(k\)-omino tower. Let \(L_j\) and \(R_j\) respectively represent the leftmost and rightmost \(k\)-omino on level \(j\) for integers \(j \geq 0\). Identify \(L_1\), the leftmost \(k\)-omino on
level one of the tower, and $L_0$, the leftmost $k$-omino in the base. Suppose the column containing
the leftmost unit square of $L_0$ does not intersect $L_1$. Then, the $k$-omino tower can be supported
by the base without $L_0$. Thus we may remove $L_0$ to obtain a $(n-1, b-1)$-k-omino tower.

Now assume $L_1$ is completely supported by $L_0$. Let $1 \leq k_0 \leq k$ be the number of columns of
which intersect both $L_1$ and $L_0$, and let $k_j$ be the number of columns which intersect both $R_j$ and
$R_{j-1}$ for $j \geq 1$. Identify the index $0 \leq j \leq k-1$ such that

1. $k_0 + k_1 + \cdots + k_j \leq k$ and
2. $k_0 + k_1 + \cdots + k_j + k_{j+1} > k$ or $k_{j+1} = 0$.

If $k_0 + k_1 + \cdots + k_j = k$, our map is defined as follows: Remove the $k$ unit squares of $L_1$, $R_1, \ldots, R_j$
enumerated by $k_0, k_1, \ldots, k_j$. These are precisely the squares supporting the blocks $L_1, R_1, \ldots, R_j$.
As the remaining $k(j+1) - k$ unit squares of these $k$-ominoes are unsupported, they fall to level
zero leaving $(b+j)k$ unit squares which can be merged and to form a base of $(b+j)$-k-ominoes.
The $k$-ominoes fully supported by $L_1$, $R_1, \ldots, R_j$ also fall leaving a $(n-1, b+j)$-k-omino tower.

Otherwise, if $k_0 + k_1 + \cdots + k_j < k$, slide the base as well as the $k$-ominoes $R_1, \ldots, R_j$ to the left
$k - (k_0 + k_1 + \cdots + k_j)$ units while all other $k$-ominoes in the tower remain fixed in relation to each
other with the base and this right staircase of $R_1, \ldots, R_j$ sliding left and under the fixed $k$-ominoes.
After the slide, the support of $L_1$ in the new tower measured by $k_0^*$ is such that $k_0^* + k_1 + \cdots + k_j = k$
and we can find a $(n-1, b+j)$-k-omino tower as described above. See Figure 3 for an illustration
of this process. This function is well-defined because if $k_{j+1} \neq 0$, the number of unit squares
supporting the block $R_{j+1}$ must be greater than $k - (k_0 + k_1 + \cdots + k_j)$ in order to satisfy the two
conditions above for identifying $k_0, k_1, \ldots, k_j$.

Further, given a fixed $(n-1, b+j)$-k-omino tower, $T$, the $(n,b)$-k-omino towers which map onto
$T$ are those with compositions $k_0 + k_1 + \cdots + k_j = k$ along with those who have slid $k_0 - 1$ ways

Figure 3: A $(7,2)$-4-omino tower, where $k_0 = 2$, $k_1 = 1$, and $k_2 = 2$, mapped to a $(6,3)$-4-omino tower.
for each composition. Thus the number of such \((n,b)\)-\(k\)-omino towers is given by

\[
\sum_{k_0+k_1+\cdots+k_j=k} k_0 = \binom{k}{j+1},
\]

where the sum is over compositions \(k_0 + k_1 + \cdots + k_j = k\) of \(k\) into \(j+1\) parts. The equality follows by a simple inductive argument where we have

\[
\binom{k}{j+1} = \binom{k-1}{j} + \binom{k-1}{j-1} = \sum_{k_0+k_1+\cdots+k_j=k-1} k_0 + \sum_{k_0+k_1+\cdots+k_{j-1}=k} k_0,\]

because compositions of \(k\) into \(j+1\) parts can be partitioned into compositions whose \((j+1)\)st part is one and compositions whose \((j+1)\)st part is greater than one. Therefore the set of \((n,b)\)-\(k\)-omino towers can be partitioned into sets which are indexed by \(0 \leq j \leq k-1\) and determined by the compositions \(k_0 + k_1 + \cdots + k_j\). Each of these sets maps onto \(\binom{k}{j+1}\) copies of the set of \((n-1,b+j)\)-\(k\)-omino towers, or equivalently, \(\binom{j}{i}\) copies of the set of \((n-1,b+i-1)\)-\(k\)-omino towers for \(1 \leq i \leq k\).

To check uniqueness, consider the inverse map which takes a \((n-1,b+j)\)-\(k\)-omino tower, \(T\), onto \(\binom{k}{j+1}\) \((n,b)\)-\(k\)-omino towers using compositions \(k_0 + k_1 + \cdots + k_j = k\) and shifts from zero to \(k_0 - 1\) squares. To apply a shift of one unit, we slide the base of the \((n,b)\)-\(k\)-omino tower and the blocks \(R_1,\ldots,R_j\) one unit to the right, and we see that the value of \(k_0\) in the new tower decreases by one which allows for a corresponding composition \((k_0-1) + k_1 + \cdots + k_j + 1 = k\) with the same shape into \(j+2\) parts. However, this implies the tower \(T\) contains the \(k\)-omino \(R_{j+1}\) whose support by \(R_j\) after the shift is one unit and hence before the shift it would not have been supported by the tower, contradicting the fact that the base is \((b+j)\) \(k\)-ominoes. Therefore the compositions \((k_0-1) + k_1 + \cdots + k_j + 1 = k\) must be associated to a \((n-1,b+j)\)-\(k\)-omino tower. The uniqueness follows similarly for shifts greater than one. Thus compositions of different sizes produce unique towers. Furthermore, two compositions of the same size must also produce different towers. The right staircase given by \(R_1,\ldots,R_j\) is unique because the sums \(k_1 + \cdots + k_j\) are unique as they represent all compositions of the integers between \(j\) and \(k-j\) into \(j\) parts.

Finally, it is left to consider the case where \(b = 1\). In this case the results above hold for the case where \(L_1\) is completely supported by \(L_0\), that is, the leftmost column of \(L_0\) intersects \(L_1\). However, we need to consider the first case. Because the number of blocks in the base is one, the recursion onto \((n-1)\)-\(k\)-omino towers with base \(b = 0\) will not correctly enumerate the \(\binom{k(n-1)-1}{n-1}\) towers described in the formula. We apply the following identity,

\[
\binom{k(n-1)-1}{n-1} = (k-1)\binom{k(n-1)-1}{n}.
\]
Thus, we need to show that set of \((n,1)\)-k-omino towers where \(L_1\) does not intersect the column containing the left end of \(L_0\) is in bijection with \((k-1)\) copies of the set of \((n-1,1)\)-k-omino towers. This is done by placing a \((n-1,1)\)-k-omino tower on a single k-omino in each of the \((k-1)\) positions so that the base of the \((n-1,1)\)-k-omino tower hangs over the new k-omino base on the right. In particular, the first two terms of the sum in Equation 3.1 give all \((2k-1)\) ways to place a \((n-1,1)\)-k-omino tower on a base of a single k-omino whereas the remaining summands count all towers with two k-ominos on the first level.

Thus, the claim is proven.

3.3 Remark. If \(k = 1\), the binomial coefficient \(\binom{n-1}{n-b}\) counts compositions of \(n\) into \(b\) nonzero parts where the integers in the composition correspond to the number of unit blocks in each column of the 1-omino tower. In this case, the total number of 1-omino towers of area \(n\) is

\[
D_1(n) = \sum_{b=1}^{n} \binom{n-1}{n-b} = 2^{n-1}.
\]

Now, Theorem 1.1 on the number of k-omino towers composed of \(n\) k-ominoes is an immediate consequence of Theorem 3.2 and the definition of the Gaussian hypergeometric function

\[
\left(\begin{array}{c}
\alpha \\
\beta
\end{array}\right)_2F_1(a, b; c; z) = \sum_{k=1}^{\infty} \frac{(a)_k(b)_k}{(c)_k} \frac{z^k}{k!}
\]

where \((x)_k\) denotes the rising Pochhammer symbol such that \((x)_0 = 1\) and

\[
(x)_n = x(x+1) \cdots (x+n-1)
\]

for integers \(n \geq 1\).

Further, Theorem 1.1 introduces an identity on hypergeometric functions

that can be generalized to an identity with parameters, \(\alpha, -\beta\) and the scaled parameter \(c = k(\alpha + \beta) + 1\) which, when \(k = 1\), is equivalent to the classical parameters, \(\alpha, \beta\) and \(c = \alpha - \beta + 1\) of Kummer’s Theorem [14].

3.4 Theorem. For \(k\alpha + k\beta + 1\) not zero or a negative integer, we have the hypergeometric identity

\[
\left(\begin{array}{c}
k\alpha + k\beta + \beta \\
\beta
\end{array}\right)_2F_1(\alpha, -\beta; k\alpha + k\beta + 1; -1) = \sum_{i \geq 0} \left(\begin{array}{c}
k\alpha + k\beta + \beta \\
\beta - i
\end{array}\right) \frac{(\alpha)_i}{i!}
\]

where \(\left(\begin{array}{c}
x \\
y
\end{array}\right)\) denotes the extended binomial coefficient \(\frac{\Gamma(x+y+1)}{\Gamma(y+1)\Gamma(x-y+1)}\).

Proof. The proof follows directly by multiplying the extended binomial coefficient, expanded in terms of the Gamma function, across the sum given by the hypergeometric function.
References


