23. Let \( A(x) = \sum_{n=0}^{\infty} a_n x^n \). Since \( a_{n+1} = 3a_n + 2^n \), \( \sum_{n=0}^{\infty} a_{n+1} x^{n+1} = \sum_{n=0}^{\infty} 3a_n x^{n+1} + \sum_{n=0}^{\infty} 2^n x^{n+1} \),
and \( A(x) - a_0 = 3xA(x) + \frac{x}{1-2x} \). Since \( a_0 = 1 \), that gives
\[
A(x)(1-3x) = 1 + \frac{x}{1-2x}
\]
\[
A(x) = \frac{1-x}{(1-3x)(1-2x)}
\]
\[
= \frac{2}{1-3x} - \frac{1}{1-2x}
\]
\[
= \sum_{n=0}^{\infty} \left( 2 \cdot 3^n - 2^n \right) x^n
\]
Thus, \( a_n = 2 \cdot 3^n - 2^n \) for all integers \( n \geq 0 \).

25. Let \( a_n \) be the number of insects after the \( n \)th year, and let \( A(x) = \sum_{n=0}^{\infty} a_n x^n \).
Since \( a_{n+1} = 2a_n + 1000 \), \( \sum_{n=0}^{\infty} a_{n+1} x^{n+1} = \sum_{n=0}^{\infty} 2a_n x^{n+1} + 1000 \sum_{n=0}^{\infty} x^{n+1} \), and
\[
A(x) - a_0 = 2xA(x) + \frac{1000x}{1-x} \). Since \( a_0 = 50 \), that gives
\[
A(x)(1-2x) = 50 + \frac{1000x}{1-x}
\]
\[
A(x) = \frac{50(1+19x)}{(1-2x)(1-x)}
\]
\[
= 50 \left( \frac{21}{1-2x} - \frac{20}{1-x} \right)
\]
\[
= \sum_{n=0}^{\infty} 50 \left( 21 \cdot 2^n - 20 \right) x^n
\]
Thus, \( a_n = 50(21 \cdot 2^n - 20) = 1050 \cdot 2^n - 1000 \) for all integers \( n \geq 0 \). At the end of the \( n \)th year, there will be \( 1050 \cdot 2^n - 1000 \) insects.

29. Let \( A(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} \). Since \( a_{n+1} = (n+1)a_n + n! \),
\[
\sum_{n=0}^{\infty} a_{n+1} \frac{x^{n+1}}{(n+1)!} = \sum_{n=0}^{\infty} (n+1)a_n \frac{x^{n+1}}{(n+1)!} + \sum_{n=0}^{\infty} n! \frac{x^{n+1}}{(n+1)!} , \text{ and } A(x) - a_0 = xA(x) + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}
\]
Since \( a_0 = 0 \),
\[ A(x) = \frac{1}{1-x} \cdot \ln\left( \frac{1}{1-x} \right) \]
\[ = (1 + x + x^2 + x^3 + \cdots) \left( x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots \right) \]
\[ = x + \left( 1 + \frac{1}{2} \right) x^2 + \left( 1 + \frac{1}{2} + \frac{1}{3} \right) x^3 + \cdots \]
\[ = \sum_{n=1}^{\infty} H_n x^n \]
\[ = \sum_{n=1}^{\infty} n! H_n \frac{x^n}{n!}, \]

where \( H_n = \sum_{i=1}^{\infty} \frac{1}{i} \) is the average number of cycles of an \( n \)-permutation (from Chapter 6, Exercise 7, page 120). Thus, \( a_n = n! H_n \), which is equal to the total number of cycles in all of the permutations in \( S_n \). To see the connection, to get an \((n+1)\)-permutation, we can insert the integer \( n+1 \) after any of the \( n \) elements of \([n]\) in an \( n \)-permutation, for \( n \cdot a_n \) cycles, or have \( n+1 \) in a singleton cycle of its own, together an \( n \)-permutation, for \( a_n + n! \) more cycles. Thus, there are a total of \( a_{n+1} = n \cdot a_n + a_n + n! = (n+1)a_n + n! \) cycles in all of the permutations in \( S_{n+1} \).

38. We use a generalization of Theorem 8.21 with 3 subsets. Let \( a_n \) be the number of ways to line up an odd number, \( n \), of people; let \( b_n \) be the number of ways to line up an even number, \( n \), of people; let \( c_n \) be the number of ways to line up \( n \) people; and let \( d_n \) be the number of ways to separate \([n]\) into disjoint subsets \( A, B, \) and \( C \), where \( |A| \) is odd and \( |B| \) is even, and then line up the elements of each subset in a line.

Then \( a_n = \begin{cases} n! & n \text{ is odd} \\ 0 & n \text{ is even} \end{cases} \), \( b_n = \begin{cases} n! & n \text{ is even} \\ 0 & n \text{ is odd} \end{cases} \), and \( c_n = n! \). The exponential generating functions are

\[ A(x) = \sum_{n=0}^{\infty} \frac{n! x^n}{n!} = x + x^3 + x^5 + \cdots = \frac{x}{1-x^2}, \]
\[ B(x) = \sum_{n=0}^{\infty} \frac{n! x^n}{n!} = 1 + x^2 + x^4 + \cdots = \frac{1}{1-x^2}, \]
\[ C(x) = \sum_{n=0}^{\infty} \frac{n! x^n}{n!} = 1 + x + x^2 + \cdots = \frac{1}{1-x}. \]

By Theorem 8.21, the exponential generating function for \( \{d_n\} \) is

\[ D(x) = A(x)B(x)C(x) = \frac{x}{(1-x)^3(1+x)^2} = \frac{1}{16} \left[ -\frac{1}{1-x} + \frac{4}{(1-x)^3} - \frac{1}{1+x} - \frac{2}{(1+x)^2} \right]. \]
Recalling that
\[ \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad \frac{2}{(1-x)^2} = \frac{d^2}{dx^2} \left( \frac{1}{1-x} \right) = \sum_{n=0}^{\infty} (n+2)(n+1)x^n, \]
\[ \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n, \quad \text{and} \quad \frac{-1}{(1+x)^2} = \frac{d}{dx} \left( \frac{1}{1+x} \right) = \sum_{n=0}^{\infty} (-1)^{n+1}(n+1)x^n, \]
we get
\[ D(x) = \frac{1}{16} \sum_{n=0}^{\infty} \left[ -1 + 2(n+2)(n+1) + (-1)^{n+1} + 2(-1)^{n+1}(n+1) \right] x^n \]
\[ = \sum_{n \text{ even}} \frac{(n+2)n}{8} x^n + \sum_{n \text{ odd}} \frac{(n+3)(n+1)}{8} x^n. \]
Thus,
\[ d_n = \begin{cases} \frac{(n+2)n \cdot n!}{8} & \text{if } n \text{ is even} \\ \frac{(n+3)(n+1)!}{8} & \text{if } n \text{ is odd} \end{cases} \]

40. Let \( a_n \) be the number of ways to arrange the \( n \) cards in a nonempty subset; then \( a_0 = 0 \) and \( a_n = n! \) for \( n > 0 \). Let \( b_n \) be the number of ways to arrange an even number, \( n \), of subsets in a line; then \( b_n = \begin{cases} n! & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases} \). Let \( g_n \) be the number of ways to split \( n \) cards into an even number of nonempty subsets, form a line within each subset, and then arrange the subsets in a line. Then, if \( A, B, \) and \( G \) are the exponential generating functions for \( \{a_n\}, \{b_n\}, \text{ and } \{g_n\} \), respectively, then
\[ A(x) = \sum_{n=1}^{\infty} \frac{n!x^n}{n!} = \frac{x}{1-x}, \quad B(x) = \sum_{n \text{ even}} \frac{n!x^n}{n!} = \frac{1}{1-x^2}, \text{ and by Theorem 8.27,} \]
\[ G(x) = B(A(x)) = \frac{1}{1 - \left( \frac{x}{1-x} \right)^2} = \frac{(1-x)^2}{(1-x)^2 - x^2} = \frac{x^2 - 2x + 1}{1 - 2x} = \frac{3}{4} - \frac{x}{2} + \frac{1}{4} \sum_{n=0}^{\infty} (2x)^n = 1 + \sum_{n=2}^{\infty} 2^{n-2} x^n \]
Thus, \( g_0 = 1, \ g_1 = 0, \) and \( g_n = 2^{n-2} n! \) for \( n \geq 2 \).

Bonus: From Example 7.4, \( d_n = \sum_{i=0}^{n} \frac{(-1)^i n!}{i!} \), so the exponential generating function is
\[ D(x) = \sum_{n=0}^{\infty} d_n \frac{x^n}{n!} = \sum_{i=0}^{n} \sum_{i=0}^{n} \frac{(-1)^i}{i!} x^n = 1 + (1-1)x + \left( 1 - 1 + \frac{1}{2!} \right) x^2 + \left( 1 - 1 + \frac{1}{2!} - \frac{1}{3!} \right) x^3 \]
\[ = \left( 1 + x + x^2 + x^3 + \ldots \right) - \left( x + x^2 + x^3 + \ldots \right) + \frac{1}{2!} \left( x^2 + x^3 + x^4 + \ldots \right) \]
\[ = \left( 1 + x + x^2 + x^3 + \ldots \right) \left( 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots \right) \]
\[ = \frac{1}{1-x} \cdot e^{-x} = \frac{e^{-x}}{1-x}. \]