CMJ 622. *Proposed by Michael Golomb, Purdue University, West Lafayette, IN*

Identical beads are distributed among the vertices of a regular *n*-gon in such a way that the center of mass of the distribution is at the center of the *n*-gon. Show that:

i. if *n* is a power of 2, the number of beads at any vertex is the same as the number on the diametrically opposite vertex;
ii. this need not be true if *n* is not a power of 2.

Solution: *By Jim Brawner, Armstrong Atlantic State University, Savannah, GA*

Let us first recall some basic facts about cyclotomic ("circle-splitting") polynomials that we will need. If *n* is a positive integer, then the cyclotomic polynomial $\Phi_n(z)$ can be defined inductively by

$\Phi_1(z) = z - 1$ and

$$\Phi_n(z) = \frac{z^n - 1}{\prod \Phi_d(z)} \ ,$$

where the product in the denominator runs over all positive divisors of *n* other than *n* itself. Notice that the degree of $\Phi_n(z)$ is given by $j(n)$, where j is the Euler j - function. It is a well-known, but by no means trivial, theorem that the cyclotomic polynomial $\Phi_n(z)$ is the irreducible or minimal polynomial in $\mathbf{Q}[z]$ for the primitive *n*th roots of unity. In other words, if w is a primitive *n*th root of unity, and $f(z)$ is a nonzero polynomial with rational coefficients such that $f(w) = 0$, then $\Phi_n(z) \big| f(z)$. In particular, the degree of *f* must be greater than or equal to the degree of $\Phi_n(z)$, which is $j(n)$. See [H, 6.5], e.g., for a proof of this theorem and an elegant discussion of cyclotomic polynomials. We now return to our regularly scheduled proof.

Let the regular *n*-gon be positioned with its center at the origin in the complex plane, and vertices at the *n*th roots of unity. Let w denote the principal *n*th root of unity $e^{2pi/n}$, so that the vertices V_k, in counter-clockwise order, are given by w^k, for $0 \le k \le n-1$. Let b_k denote the number of beads at the vertex V_k, so that the condition that the center of mass of the distribution is at the center of the *n*-gon is given by

$$\sum_{k=0}^{n-1} b_k w^k = 0 \, ,$$

where the b_k are nonnegative integers.

i) If $n = 2^0 = 1$ then the *n*-gon is simply a point, which can be thought of as being diametrically opposite to itself. If *n* is an *even* power of 2, say $n = 2^a$, $a \ge 1$, then the vertex diametrically opposite V_k, $0 \le k \le n/2$, is $V_{k+n/2}$, and

$$w^{k+n/2} = w^k e^{(2pi/n)(n/2)} = -w^k \, .$$

The condition on the center of mass can then be reformulated as

$$\sum_{k=0}^{n/2-1} (b_k - b_{k+n/2}) w^k = 0 \, .$$

This means that $f(z) = \sum_{k=0}^{n/2-1} (b_k - b_{k+n/2}) z^k$ is a polynomial with rational (in fact, integer) coefficients such that $f(w) = 0$. If $f(z)$ is nonzero, then it has degree less than *n*/2.

$0 \le k \le n/2$

But the Euler j -function for $n = 2^a$ is $j(2^a) = 2^{a-1} = \frac{n}{2}$, which is the degree of the irreducible polynomial in $\mathbf{Q}[z]$ for the principal *n*th root of unity w. Therefore, $f(z)$ must be identically zero, by the theorem mentioned above. Thus, $b_k = b_{k+n/2}$ for $0 \le k \le n/2$, and the number of beads at any vertex is the same as the number on the opposite vertex.

ii) If $n > 1$ is odd, then no vertex has a diametrically opposite vertex, so the statement is meaningless. If *n* is even, but not a power of 2, then $n = 2^a r$, where $r \ge 3$ is odd and $a \ge 1$. For $0 \le k \le n-1$, let

$$b_k = \begin{cases} 1 & \text{if } 2^a | k \\ 0 & \text{otherwise} \end{cases} .$$

Then $b_0 = 1$, but $b_{n/2} = 0$, since $\frac{n}{2} = 2^{a-1} r$, and 2^a does not divide $2^{a-1} r$. Therefore, the number of beads at vertex V_0 is not the same as the number of beads on the diametrically opposite vertex $V_{n/2}$.

On the other hand, if we set $s = 2^a$, then

$$\sum_{k=0}^{n-1} b_k w^k = 1 + w^s + \left(w^s\right)^2 + \cdots + \left(w^s\right)^{r-1} \qquad (*)$$

where $w^s = e^{(2pi/n)s} = e^{2pi/r}$ is a principal *r*th root of unity, and hence a zero of $z^r - 1 = (z-1)(1 + z + z^2 + \cdots + z^{r-1})$. Since $r > 1$, $w^s \ne 1$, so w^s must be a zero of $1 + z + z^2 + \cdots + z^{r-1}$, and $\sum_{k=0}^{n-1} b_k w^k = 0$ by Equation (*) above.

For illustration, the simplest case is when *n* = 6, where $(b_0, b_1, b_2, b_3, b_4, b_5)$ = (1,0,1,0,1,0), which gives the hexagon in Figure 1 below:

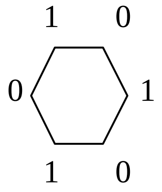

Figure 1

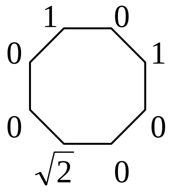

Figure 2

Note: Statement (i) would not have been true if we had been allowed to use beads of arbitrary (real number) weights. This follows from the fact that the cyclotomic polynomial is irreducible in **Q**[z], but not necessarily in **R**[z]. For example, if $n = 8$, then $\Phi_8(z) = z^4 + 1 = (z^2 - \sqrt{2}z + 1)(z^2 + \sqrt{2}z + 1)$, and $w = e^{2pi/8}$ is a zero of $f(z) = z^2 - \sqrt{2}z + 1$, which leads to the distribution $(1,0,1,0,0,\sqrt{2},0,0)$ in Figure 2.

References

[H] I.N. Herstein, *Abstract Algebra,* 2nd edition, Macmillan Publishing Co., New York, 1990.

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